Chapter 5

Statistical fluctuation of repulsive condensate in anharmonic trap

Recent experimental achievements suggest that, transition temperature increases significantly if one makes the trap tight [147, 148, 149, 150]. So, the effect of a tight trap on the condensation is more pronounced. To corroborate with recent experimental observation the attention has now shifted to the anharmonic trap of finite extent over the usually used harmonic oscillator trap of infinite extension. An anharmonic confinement can be modelled as $V(r) = \frac{1}{2}m\omega^2 x^2 + \gamma r^4$, where γ is a controllable parameter- tunable in controlled fashion by changing the laser frequency. For $\gamma > 0$ then the frequency is blue-shifted and the trap becomes more tighter and steper at large distances from the centre of the atom cloud. For $\gamma < 0$ the frequency is red-shifted and how the effective potential changes depending on the value and sign of γ value has been discussed in many earlier works [41, 151, 152]. For $\gamma < 0$ the trapping potential can be closely approximated to be a harmonic potential with a Gaussian envelope as $V = V_0 r^2 e^{-\frac{\gamma r^2}{2}}$. It affects the stability of the condensate in a peculiar manner unlike the usual harmonic trap with negative scattering length $a_s < 0$ [39]. Several multipole frequencies have already studied but no detailed study of fluctuation in the ground state occupation number and several orders of central moment of the fluctuation for the particles in anharmonic trap have not been addressed yet. For our present study, we have restricted ourselves only for $\gamma > 0$ case. Repulsively interacting ${}^{87}Rb$ condensate is chosen as our system for the study of effect of anharmonic trapping

on several statistical fluctuations. The system parameters mimic the JILA trap experiment.

5.1 Ground state probability distribution

We studied the variation of ground state probability distribution (P_{n_0}) as a function of ground state occupation number n_0 for two anharmonicities $\gamma = 10^{-6}$ and 10^{-4} . In figure 5.1, we presented distribution function for N = 3000 repulsively interacting trapped bosons in an anharmonic trap for different set of temperatures $viz., T = 0.2T_c, 0.5T_c$ and $0.8T_c$. We observe that when the temperature increases the peak value of the ground state probability distribution (P_{n_0}) decreases. It shows maximum for very small temperature $T = 0.2T_c$ near the transition temperature T_c . At higher temperatures, the interparticle repulsion pushes the atoms to higher excited states. The population depletion in the ground state happens also the population in the excited state increases. The comparison is made with



Figure 5.1: Plot of ground state probability distribution (P_{n_0}) for N = 3000 atoms of ⁸⁷Rb for different temperatures $(T = 0.2T_c, 0.5T_c \text{ and } 0.8T_c)$ both in harmonic and anharmonic trap.

the harmonic oscillator trap for which $\gamma = 0$. The peak of P_{n_0} corresponding to strong anharmonicity $\gamma = 10^{-4}$ shows higher value in comparison with both smaller anharmonicity $\gamma = 10^{-6}$ and harmonic trap. Also the distribution for the anharmonic trapping are centered at a larger n_0 value in comparison with the harmonic trapping. These two observations support the conclusion that anharmonic trap enhances the condensation process - more and more number of particles will accumulate in the ground state as the anharmonicity γ increases from 10^{-6} to 10^{-4} .

5.2 Condensate fraction

The thermal evolution of the condensate fraction for repulsively interacting ${}^{87}Rb$ atoms, trapped in an anharmonic trap, is shown in the following figure 5.2. For each specific calculation we fix the atom number N = 3000 and the reduced temperature $T = 0.2T_c$, $0.5T_c$ and $= 0.8T_c$. The qualitative behaviour is the same



Figure 5.2: Plot of condensate fraction against the reduced temperature T/T_c for indicated numbers of interacting atoms of ${}^{87}Rb$ both in the harmonic and anharmonic confinement.

for the two different anharmonicity $\gamma = 10^{-6}$ and 10^{-4} . But the possibility of achieving high-temperature BEC is more favourable in a strong anharmonic trap as the condensate fraction curve is shifted to the right side. The comparison is made with the repulsive interacting harmonically trapped BEC. All the curves show a smooth transition near the critical point. However, at large anharmonicity $\gamma = 10^{-4}$, we observe the emergence of the more populated ground state. As the

anharmonicity increases the achievement of high-temperature BEC becomes more possible.

5.3 Central moments of the fluctuation and characteristic temperatures

Several orders of central moment of fluctuation as a function of reduced temperature $\frac{T}{T_c}$ for the repulsive interacting BEC in anharmonic trap have been discussed. In chapter 4, we have discussed the behaviour of these moments for attractive interacting Bose gas trapped in a harmonic oscillator and comparison was made with the noninteracting case. The effect of interaction on the measured quantities were discussed. However, here we will see the effect of anharmonicity γ on the moments of the fluctuation. The normalized second-order central moment $\frac{\langle (n_0 - \langle n_0 \rangle) \rangle^2}{N}$, usually termed as the standard deviation (σ), is plotted in figure 5.3. The behaviour of the moment is like a lambda-structure. We see that σ is zero at absolute zero temperature and smoothly reaches the maximum value around the transition temperature then suddenly drops to zero. The sharp change of the standard deviation can be considered as a signature of phase transition and the transition temperature corresponds at the peak of the curve.

The normalized third-order central moment $\frac{\langle (n_0 - \langle n_0 \rangle) \rangle^3}{N}$ is plotted as a function of the reduced temperature $\frac{T}{T_c}$ in figure 5.4. The repulsive interaction stimulates BEC as it yields an increase in the average condensate particle number. This is described as the contraction in the momentum space as the particles try to occupy several other excited states due to repulsive interaction which pushes the atoms to the excited states. As the effective interaction increases due to the anharmonic trapping, further enhancement in the BEC process emerges. The peak value for the third-order central moment is higher for the stronger anharmonic trap with anharmonicity $\gamma = 10^{-4}$. The result for harmonic oscillator is also presented for comparison. The peak value of the third-order central moment for harmonic trap is smaller than both the anharmonic trapping with anharmonicity parameters $\gamma = 10^{-6}$ and 10^{-4} . So, we can say that anharmonic trapping not only favours high-temperature BEC but also supports the possibility of phase transition for



Figure 5.3: Plot of standard deviation against the reduced temperature T/T_c for indicated number of interacting atoms of ${}^{87}Rb$ both in the harmonic and anharmonic confinement.



Figure 5.4: Third-order central moment against the reduced temperature T/T_c is plotted for indicated number of interacting atoms of ${}^{87}Rb$ both in the harmonic and anharmonic confinement.

interacting system in mesoscopic regime. The transition temperature T_c can be calculated from the peak position. Similarly, plot of the fourth-order central moment $\frac{\langle (n_0 - \langle n_0 \rangle) \rangle^4}{N}$ with respect to reduced temperature $\frac{T}{T_c}$ is shown in figure 5.5.



Figure 5.5: Fourth-order central moment against the reduced temperature T/T_c is presented for indicated number of interacting atoms of ${}^{87}Rb$ both in the harmonic and anharmonic confinement.

This moment also smoothly increases up to a specific temperature then drops to zero suddenly - standard lambda-type-structure is evident. Inflexion point of the lambda type of structure near the transition point in all the measures has been observed. The shift of the inflexion point of the lambda-structure of every moment to the right side with an increase in the anharmonicity parameter γ proves that the effect of anharmonicity enhances the possibility of phase transition. We can define three characteristic temperatures from the inflexion point of the lambda-structures corresponding to the second-, third- and fourth-order central moments and their dependence on anharmonicity. We introduce three characteristic temperatures T_{σ} , $T_{3-moment}$ and $T_{4-moment}$ from the inflexion point of the lambda-structure of standard deviation, third- and fourth-order central moments respectively. In Table 5.1, we present the dependence of these three characteristic temperatures on anharmonicity. We observe that they increase with increase in anharmonicity which jointly confirms that BEC is more favourable in anharmonic trap.

We checked our calculations for smaller as well as for larger particle number. The general trend remains the same, however, the finite-sized effect is seen in the curvature of the lambda-structures. For smaller particle number, we observe very broad maxima at the transition point, whereas for larger particle the sharpness of

Table 5.1: Characteristic temperatures calculated from different fluctuation measures and for different anharmonicity with N = 3000. T_{σ} , $T_{3-moment}$ and $T_{4-moment}$ are defined in the text.

Anharmonicity (γ)	T_{σ}	$T_{3-moment}$	$T_{4-moment}$
0	0.857	0.875	0.854
10^{-6}	0.917	0.936	0.915
10 ⁻⁴	1.209	1.238	1.199

the lambda structure increases. Of course, it supports the true phase transition in thermodynamic limit, but not for the mesoscopic condensate. We do not find any possibility of true phase transition in anharmonic trap, although it favours the achievement of BEC at a higher temperature. Thus, the calculation of statistical fluctuations of the condensate in anharmonic trap opens the further study of the possibility of quasi phase transition instead of true phase transition.

5.4 Transition exponent for quasi phase transition

The critical transition point is generally characterized by the divergence in the thermodynamic quantities. Thus, at the critical point, the thermodynamic quantities loose their analytic behavior. In terms of reduced temperature $t = \frac{T-T_C}{T_C}$, we can define the critical exponent λ_c for any thermodynamic function F(t) as

$$\lambda_c = \lim_{t \to 0} \frac{\ln |F(t)|}{\ln |t|}.$$
(5.1)

Thus for small |t|, $F(t) \sim t^{\lambda_c}$. At the critical temperature, F(t) may either vanish or becomes singular. However, for quasi phase transition, the thermodynamic quantities are not singular, instead they smoothly vary across T_c . In this case, following the prescription of Ref. [153], F(t) can be written as $F(t) = F(0)+c|t|^{\lambda_c}+...$, thus λ_c is the leading exponent and determines the nature of thermodynamic function near the critical point. For quasi phase transition λ_c is called as transition exponent and is given by Eq. 5.1. It is to be noted that for the phase transition, λ_c exhibits universality, does not depend on the details of system parameters, only depends on the dimension of the system. In the present work, we calculate the transition exponent from utilizing the specific heat capacity $C_N(T)$. The required formula for the calculation of the specific heat is given in Eq. 3.20 in Chapter 3.3. Discontinuity in the specific heat gives the second order phase transition. But



Figure 5.6: Plot of specific heat against the reduced temperature $\frac{T}{T_c}$ for N = 3000 interacting atoms of ${}^{87}Rb$ both in the harmonic and anharmonic confinement.

around the transition temperature of BEC specific heat changes smoothly so we observe quasi phase transition. In figure 5.6, we plot $\frac{C_N(T)}{Nk_B}$ as a function of reduced temperature $\frac{T}{T_c}$ for N = 3000 bosons both in the harmonic as well as anharmonic trap. It is clearly seen that T_c increases smoothly with anharmonicity as expected and condensation is more favourable in anharmonic trap. In general, the transition temperature T_c is calculated at the maximum of specific heat capacity

$$\frac{\partial C_N(T)}{\partial T}\Big|_{T_c} = 0.$$
(5.2)

To calculate the transition exponent, we utilize the expression, Eq. 5.1, separately

for $T < T_c$ and $T > T_c$ and redefined for the two regions as

$$\lambda_c^{\pm} = \lim_{t \to 0\pm} \frac{\ln |F(t) - F(0)|}{\ln |t|}.$$
(5.3)

In figure 5.7- 5.9 we plot $\ln \left| \frac{C_N(T) - C_N(T_c)}{Nk_B} \right|$ as a function of $\ln |t|$ for several γ . In



Figure 5.7: Transition exponent fitting for harmonic trap.



Figure 5.8: Transition exponent fitting for anharmonic trap $\gamma = 10^{-06}$.

panel (a) of each of figure 5.7- 5.9 we present our results for t < 0 and make a straight line fit identified as $|t|^{\lambda_c}$. The slope of these lines are 1.561 for harmonic, 1.771 for anharmonicity 10^{-6} and 1.793 for anharmonicity 10^{-4} . Thus, we note that transition exponent smoothly increases in anharmonic trap. In panel (b) of figure 5.7- 5.9, we plot the same for t > 0. The corresponding transition exponents



Figure 5.9: Transition exponent fitting for anharmonic trap $\gamma = 10^{-04}$.

are 2.671 for harmonic, 2.312 for an harmonicity 10^{-6} and 2.282 for an harmonicity $10^{-4}.\,$

We observe that $\Delta \lambda_c = \lambda_c(t > 0) - \lambda_c(t < 0)$ is maximum for the harmonic trap and decreases for anharmonic traps. We conclude that change in $C_N(T)$ near the transition temperature is sharper in harmonic trap - the quasi phase transition is more favourable in this trap.