Chapter 1

Introduction

"But man is not made for defeat...

A man can be destroyed but not defeated." - Ernest Hemingway

Algebraic graph theory is the part of mathematics where we use algebraic techniques to solve graph-theoretic problems. We try to convert the properties of a graph into algebraic properties and use results of algebra to solve them. Therefore the fundamental aim is by using tools of algebra, to analyze the behaviour of the graph and to prove interesting results.

There are several mathematical ways including group theory, matrix theory, probability theory, combinatorics to handle a problem of graph theory. In the branch of spectral graph theory, we study the structure and properties of graph with the help of linear algebra and matrix theory. Adjacency matrix, incidence matrix, Laplacian matrix, eigen values in the spectrum reveals the construction of the graph and we use them to study different graph parameters.

There is another branch of algebraic graph theory where people are interested to define graphs on different algebraic structures like group, ring, module etc. The idea of associating a graph with a group originates with Cayley graph and now it has become an interesting topic for many researchers. The most prominent graphs that are defined on groups in recent years are power graph, enhanced power graph, commuting graph, non commuting graph, subgroup inclusion graph etc has been done in this topic. As a comprehensive survey on different graphs defined on groups [18], is an excellent reference. These graphs help us in understanding various group properties using graph theoretic interpretation and vice-versa.

1.1 Overview of Chapters

The contents of this thesis is divided into five chapters including this chapter. In chapter one we provide an introduction and motivation towards the topic of this thesis. The overview of other chapters are briefly discussed in the following orientation:

Chapter 2: In this chapter we start with a graph which is defined on a group namely co-maximal subgroup graph, denoted by $\Gamma(G)$. The vertices of $\Gamma(G)$ are non-trivial proper subgroups of G and two distinct vertices H and K are adjacent if and only if $HK = G$. Since isolated vertices may occur in $\Gamma(G)$, we try to characterise the subgroups that correspond to isolated vertex and prove some results on connectedness. We focus on understanding the structural properties of $\Gamma(G)$ and $\Gamma^*(G)$ like completeness, bipartiteness, girth, universal vertex etc. In the next section, we try to find independence number, chromatic number, perfectness of $\Gamma(G)$. We have stressed upon the result of independence number of $\Gamma(G)$ as it gives information about the underlying group structure.

We also try to classify the groups that give isomorphic co-maximal subgroup graphs. Finally we conclude this chapter with some open issues to the reader.

Chapter 3:

This chapter deals with co-maximal subgroup graph for two particular family of groups, namely Z_n and D_n. Staring with some basic properties of $\Gamma(\mathbb{Z}_n)$ and $\Gamma^*(\mathbb{Z}_n)$, we prove some results on connectedness and diameter. In the consecutive sections, hamiltonicity, dominating sets, perfectness of $\Gamma(\mathbb{Z}_n)$ are explored. We discuss the conditions under which co-maximal subgroup graphs defined over different cyclic groups are isomorphic.

We also characterize various structural properties of $\Gamma(D_n)$ and $\Gamma^*(D_n)$. Further we study the domination number, chromatic number of $\Gamma(D_n)$ and characterize when $\Gamma(D_n)$ is perfect. Results of next section claims that nilpotent dihedral groups are uniquely determined by their co-maximal subgroup graphs.

Chapter 4:

In this chapter, we define a new graph associated with the ideals of a commutative ring R, called prime ideal sum graph of a commutative ring R, denoted by $PIS(R)$ and highlight the connections between algebraic properties of a ring and the graph-theoretic properties of its graph.

In subsequent sections, we discuss connectedness, completeness, universal vertex, isolated vertices in $PIS(R)$. We also investigate girth and domination number of $PIS(R)$. Moreover, the results show strong connection between girth of a graph and prime ideals of R. Then we discuss some properties related to homomorphisms and isomorphisms of $PIS(R)$. The results indicates the effect of homomorphism (isomorphism) between rings R and S on the homomorphism (isomorphism) between $PIS(R)$ and $PIS(S)$. Finally the prime ideal graph $PIS(R)$ is studied for general ZPI-Rings.

Chapter 5:

This chapter deals with the condition of perfectness of another family of graph, that is again associated with ring, called annihilating ideal graph of Z_n. The main result in this chapter involves the study of characterizing n for which $AG(Z_n)$ is perfect.

Chapter 6: In this chapter, we summarize and conclude the thesis with some direction for future research.