Chapter 2

Comaximal Subgroup Graph of a Group

2.1 Introduction

In this chapter, we mainly focus on various aspects of graphs defined on a group structure and discuss many important connections between them. The idea of associating graphs with groups, which started from Cayley graphs, is now an important topic of research in modern algebraic graph theory. The most prominent graphs defined on groups in recent years are power graphs [19], enhanced power graphs [1], commuting graphs [33], non-commuting graphs [2], subgroup inclusion graphs [25] etc., and various works like [15] has been done on these topics. As a comprehensive survey on different graphs defined on groups, [18] is an excellent reference. These graphs help us in understanding various group properties using graph theoretic interpretations. Following these footsteps, Akbari *et.al.* introduced *co-maximal subgroup graph* of a group G in [6].

Definition 2.1.1 *Let G be a group and S be the collection of all non-trivial proper subgroups of G. The co-maximal subgroup graph $\Gamma(G)$ of a group G is defined to be a graph with S as the set of vertices and two distinct vertices H and K are adjacent if and only if $HK = G$.*

Although the definition of subgroup product graph allows the possibility of G being infinite, in this thesis, we restrict ourselves to finite groups only. However if the results translate similarly to infinite groups, we will mention it separately. Note that the definition implies that the graph is undirected as $HK = G$ if and only if $KH = G$. In this chapter, we continue the study of co-maximal subgroup graph of a group. We also introduced deleted co-maximal subgroup graph $\Gamma^*(G)$ which is obtained by removing the isolated vertices of $\Gamma(G)$. We study the existence of isolated vertices of $\Gamma(G)$, connectedness of $\Gamma^*(G)$ and characterize various properties of $\Gamma(G)$ and $\Gamma^*(G)$.

2.1.1 Preliminaries

We first recall some definitions and results on graph theory and group theory. For undefined terms and results on group theory, please refer to [16] and [37] and that of graph theory, please refer to [44].

Let Γ be a graph. The *diameter* of a connected graph Γ is the maximum distance between any two vertices in Γ. The minimum degree of a vertex in Γ is denoted by $\delta(\Gamma)$ and $girth(\Gamma)$ denotes the length of a smallest cycle in Γ.

Let G be a finite group. Denote by $\pi(G)$, the set of prime divisors of $|G|$. A proper subgroup H of a group G is said to be a *maximal subgroup* if there does not exist any proper subgroup of G which properly contains H. A group G is said to be *minimal non-cyclic* if G is non-cyclic but every proper subgroup of G is cyclic. The set of all maximal subgroups of the group G is denoted by $Max(G)$. The *Frattini subgroup* of a group G is defined as the intersection of all maximal subgroups of G and is denoted by $\Phi(G)$. The *intersection number* of a finite group G, denoted by $\iota(G)$, is the minimum number of maximal subgroups of G whose intersection is equal to $\Phi(G)$. By D_{2n}, we mean the dihedral group of order $2n$.

Now, we state a few standard group theoretic results which we will be using throughout the paper.

Theorem 2.1.1 (Miller and Moreno, 1903 ([32],[43],Theorem 1)) *A group G is a minimal non-cyclic group if and only if G is isomorphic to one of the following groups:*

1. $\mathbb{Z}_p \times \mathbb{Z}_p$.
2. *The quaternion group Q_8 of order* 8
3. $\langle a, b | a^p = b^{q^m} = 1, b^{-1}ab = a^r \rangle$, *where p and q are distinct primes and* $r \not\equiv 1 \ (mod\ p), r^q \equiv 1 \ (mod\ p)$.

Proposition 2.1.1 *Let G be a finite group.*

1. *If G has a unique maximal subgroup, then G is cyclic p-group.*

2. *If G has exactly two maximal subgroups, then G is cyclic and $|G| = p^a q^b$, where p, q are distinct primes.*

3. *G is nilpotent if and only if all maximal subgroups of G are normal in G.*

4. *$|G/\Phi(G)|$ is divisible by all primes p dividing $|G|$.*

2.1.2 Previous Works

In [6], authors proved various results on $\Gamma(G)$. We recall a few of them, which will be used in this paper.

Theorem 2.1.2 ([6], Theorem 2.2) *Let G be a group. If $\delta(\Gamma(G)) \geq 1$, then $diam(\Gamma(G)) \leq 3$.*

Theorem 2.1.3 ([6], Theorem 2.4) *Let G be a finite group with at least two proper non-trivial subgroups. Then the following are equivalent:*

1. *$\Gamma(G)$ is connected.*

2. *$\delta(\Gamma(G)) \geq 1$.*

3. *G is supersolvable and its Sylow subgroups are all elementary abelian.*

4. *G is isomorphic to a subgroup of a direct product of groups of squarefree order.*

Corollary 2.1.2 *[6] Let G be a nilpotent group. Then $\Gamma(G)$ is connected if and only if $\Phi(G) = \{e\}$ or $G \cong \mathbb{Z}_{p^2}$, for some prime number p.*

Theorem 2.1.4 *[[6], Theorem 3.5] Let G be a nilpotent group. Then the following are equivalent.*

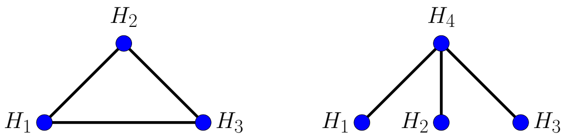
1. *There exists a vertex adjacent to all other vertices of $\Gamma(G)$.*
2. *$G \cong \mathbb{Z}_p \times \mathbb{Z}_q$, where p and q are (not necessarily distinct) primes.*
3. *$\Gamma(G)$ is complete.*

2.2 Isolated Vertices and Connectedness of $\Gamma(G)$

In [6], authors mainly focussed on graphs $\Gamma(G)$ with $\delta(\Gamma(G)) \geq 1$, i.e., graphs without isolated vertices. The only discussion on isolated vertices appear in Remark 2.9 [6], where they characterized isolated vertices in the case when G is abelian.

We start with some examples of $\Gamma(G)$, both connected and disconnected.

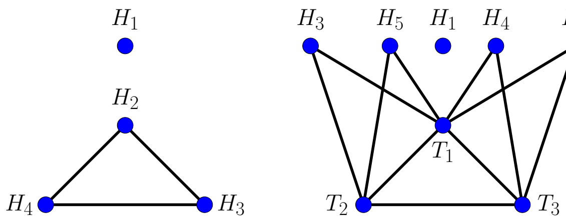
Example 2.2.1 *Consider the Klein-4 group, K_4. Then the set of all proper nontrivial subgroups is $S = \{H_1 = \{e,a\}, H_2 = \{e,b\}, H_3 = \{e,ab\}\}$ and the corresponding $\Gamma(K_4)$ is given in Figure 2.1(A). Next, consider the group S_3. Then the set of all proper nontrivial subgroups is $S = \{H_1 = \{e,(12)\}, H_2 = \{e,(13)\}, H_3 = \{e,(23)\}, H_4 = \{e,(123),(132)\}\}$ and the corresponding $\Gamma(S_3)$ is given in Figure 2.1(B).*



(A) $\Gamma(K_4)$ **(B)** $\Gamma(S_3)$

Figure 2.1: Examples of $\Gamma(G)$ (Connected Examples)

Example 2.2.2 *Consider the Quaternion group,* $Q_8 = \langle a, b : a^4 = e, a^2 = b^2, ba = a^3b \rangle$*. Then the set of all proper nontrivial subgroups is* $S = \{H_1 = \langle a^2 \rangle, H_2 = \langle a \rangle, H_3 = \langle ab \rangle, H_4 = \langle b \rangle\}$ *and the corresponding* $\Gamma(Q_8)$ *is given in Figure 2.2(A). Next, consider the Dihedral group* $D_8 = \langle a, b : a^4 = e, b^2 = e, ba = a^3b \rangle$*. Then the set of all proper nontrivial subgroups is* $S = \{H_1 = \langle a^2 \rangle, H_2 = \langle b \rangle, H_3 = \langle ab \rangle, H_4 = \langle a^2b \rangle, H_5 = \langle a^3b \rangle, T_1 = \langle a \rangle, T_2 = \{e, a^2, b, a^2b\}, T_3 = \{e, ab, a^2, a^3b\}\}$ *and the corresponding* $\Gamma(D_8)$ *is given in Figure 2.2(B).*



(A) $\Gamma(Q_8)$ **(B)** $\Gamma(D_8)$

Figure 2.2: Examples of $\Gamma(G)$ (Disconnected Examples)

□

Theorem 2.2.1 *Let* G *be a finite group. If* $\{e\} \subsetneq H \subseteq \Phi(G)$*, then* H *is an isolated vertex in* $\Gamma(G)$*. Conversely, if* G *is nilpotent and* H *is an isolated* *vertex in* $\Gamma(G)$*, then* $H \subseteq \Phi(G)$.

Proof : Let H be a non-trivial proper subgroup of G which is contained in $\Phi(G)$. If possible, let $H \sim K$ in $\Gamma(G)$ for some non-trivial proper subgroup K of G. Then there exists a maximal subgroup M of G which contains K. Thus, $G = HK \subseteq HM = M \neq G$, a contradiction. Thus H is an isolated vertex in $\Gamma(G)$.

Conversely, let H be an isolated vertex in $\Gamma(G)$. If possible, let M be a maximal subgroup of G which does not contain H. As G is nilpotent, by Proposition 2.1.1(3), M is a normal subgroup of G and hence HM is a subgroup of G and $M \subsetneq HM$. Thus, by maximality of M, we have $HM = G$, i.e., $H \sim M$ in $\Gamma(G)$, a contradiction. Thus H is contained in every maximal subgroup of G, i.e., $H \subseteq \Phi(G)$. □

Remark 2.2.1 *The above theorem proves that if the Frattini subgroup of* G *is non-trivial, then* $\Gamma(G)$ *is disconnected. It is to be noted that if* G *is not nilpotent, triviality of the Frattini subgroup of* G *does not imply* $\Gamma(G)$ *is connected. For example,* A_4 *is not nilpotent and* $\Phi(A_4)$ *is trivial. But* $\Gamma(A_4)$ *is the disjoint union of a star* $K_{1,4}$ *and three isolated vertices. This shows that solvability of* G *is not enough for the converse to hold. We can even say something more: super-solvability is also not enough. Consider the Frobenius group* $G = \langle a, b : a^5 = b^4 = 1, ab = ba^2 \rangle$ *of order* 20. *It is super-solvable, non-nilpotent group with* $|\Phi(G)| = 1$*, but it has five isolated vertices.*

Theorem 2.2.2 *Let G be a finite group. If G is a cyclic p-group, then $\Gamma(G)$ has no edges. Conversely, if G is a solvable group such that $\Gamma(G)$ has no edges, then G is a cyclic p-group.*

Proof : If G is a cyclic p-group, then $G \cong \mathbb{Z}_{p^k}$ and $S = \{\langle p \rangle, \langle p^2 \rangle, \cdots, \langle p^{k-1} \rangle\}$ is the set of vertices of $\Gamma(G)$. Clearly, all of the vertices are isolated in $\Gamma(G)$.

For the other direction, since G is a solvable group, by a Theorem of Hall (See [37], Theorem 5.28, pp. 108), G is the Zappa-Szep product of a Sylow p-subgroup H and a Hall p'-subgroup K, i.e., $G = HK$ and $H \cap K = \{e\}$. If K is non-trivial, then we have an edge (H, K) in $\Gamma(G)$. Thus K must be trivial. Hence G is a finite p-group of order, say p^n. By Sylow's theorem, G has a subgroup N of order p^{n-1} and N is normal in G. Let $g \in G \setminus N$ and $A = \langle g \rangle$. If A is a proper subgroup of G, then AN is a subgroup of G containing N and $p^{n-1} = |N| < |AN|$, i.e., $|AN| = p^n$, i.e., $AN = G$. Thus we get an edge (N, A) in $\Gamma(G)$, a contradiction. Hence $A = G = \langle g \rangle$, i.e., G is a cyclic p-group. □

Remark 2.2.2 *The solvability of G is required for the converse to hold: If $G = PSL(2, 13)$, then $\Gamma(G)$ is edgeless. In fact, there are more groups for which $\Gamma(G)$ is edgeless. In [28] authors shows that 15 of the 26 simple sporadic groups does not admit a factorization. Thus for all these 15 groups, co-maximal subgroup graph is edgeless. In the next theorem, we characterize G for which $\Gamma(G)$ is edgeless.*

Lemma 2.2.3 *Let G be a finite group such that $\Gamma(G)$ is edgeless. If $\Phi(G)$ is trivial, then G is simple.*

Proof : If possible, let H be a non-trivial proper normal subgroup of G. As $\Phi(G)$ is trivial, there exists at least one maximal subgroup M of G which does not contain H. Thus M is adjacent to H in $\Gamma(G)$, a contardiction. Hence G is simple. □

Theorem 2.2.3 *Let G be a finite group. Then $\Gamma(G)$ is edgeless if and only if $G/\Phi(G)$ is simple and has no factorization.*

Proof : $\Gamma(G)$ is edgeless if and only if G has no factorization if and only if $|G| > |HK|$ for any two maximal subgroups H and K of G, i.e., $[G : H] > [K : H \cap K]$ for any two maximal subgroups H and K of G. Now, as $\Phi(G)$ is contained in all maximal subgroups of G, we have G has no factorization into maximal subgroups if and only if $G/\Phi(G)$ has no factorization, i.e, $\Gamma(G/\Phi(G))$ is edgeless. Also, as $G/\Phi(G)$ has trivial Fratinni subgroup, from the previous lemma, we get $G/\Phi(G)$ is simple.

Conversely, let $G/\Phi(G)$ is simple and has no factorization. If possible, let H be adjacent to K in $\Gamma(G)$. Without loss of generality, we can take both H and K to be maximal subgroups of G. Then $\Phi(G) \subseteq H, K$ and hence $(H/\Phi(G))(K/\Phi(G)) = G/\Phi(G)$ is a factorization of $G/\Phi(G)$, a contradiction. □

In Theorem 2.2 of [6], the authors proved that if $\Gamma(G)$ has no isolated vertices, then it is connected and its diameter is bounded above by 3. In the next theorem, we discuss about the components of $\Gamma(G)$, if $\Gamma(G)$ has isolated vertices.

Theorem 2.2.4 *Let G be a finite group such that G has a maximal subgroup which is normal in G. Then $\Gamma(G)$ is connected apart from some possible isolated vertices. Moreover, the diameter of the component is less than or equal to 4.*

Proof : If G is a cyclic p-group, then $\Gamma(G)$ is edgeless and hence the result holds. So, we assume that G is not a cyclic p-group. Let H be a maximal subgroup of G which is normal in G. As G is not a cyclic p-group, there exists another maximal subgroup L of G. Thus $HL = G$ and $H \sim L$, i.e., H is not an isolated vertex in $\Gamma(G)$. Let C_1 be the component of $\Gamma(G)$ which contains H. If possible, let there exists another component C_2 of $\Gamma(G)$ and $H' \sim K'$ in C_2.

If both $H', K' \subseteq H$, then $H'K' \subseteq H \neq G$, which contradicts that $H' \sim K'$. Thus, at least one of H' and K' is not contained in H. Let H' is not contained in H. Then $H \subsetneq HH' = G$ and hence $H \sim H'$ in $\Gamma(G)$. This contradicts that H and H' are in different component of $\Gamma(G)$. Hence, our assumption is wrong and $\Gamma(G)$ is connected apart from some possible isolated vertices.

Now, we prove the upper bound for the diameter. If H is the only maximal subgroup of G, then G is a cyclic p-group, and the resulting graph $\Gamma(G)$ is empty. So, we assume that there exist other maximal subgroups of G, apart from H. Let A, B be two arbitrary vertices of the component. If $A, B \not\subseteq H$, then we have $A \sim H \sim B$, i.e., $d(A, B) \leq 2$.

If $A \not\subseteq H$ and $B \subseteq H$, then as B is not an isolated vertex in the component, there exists a subgroup B' of G such that $B' \sim B$ in $\Gamma(G)$. Clearly B' is contained in some maximal subgroup M of G, with $M \neq H$. Thus $B \sim M \sim H \sim A$, i.e., $d(A, B) \leq 3$.

Lastly, let us assume $A, B \subseteq H$. Clearly $A \nsim B$ in $\Gamma(G)$. As A and B are not isolated vertices in the component, there exist subgroups A' and B' such that $A \sim A'$ and $B \sim B'$ in $\Gamma(G)$. Again, A' and B' are contained in some maximal subgroups M_A and M_B respectively, where $H \neq M_A, M_B$. If $M_A = M_B$, then $A \sim M_A \sim B$, i.e., $d(A, B) = 2$. If $M_A \neq M_B$, then we have $A \sim M_A \sim H \sim M_B \sim B$, i.e., $d(A, B) \leq 4$. □

Corollary 2.2.4 *Let G be a finite nilpotent group. Then $\Gamma(G)$ is connected apart from some possible isolated vertices and the component has diameter at most* 3.

Proof : As every maximal subgroup in a finite nilpotent group G is normal in G, by Theorem 2.2.4, $\Gamma(G)$ is connected apart from some possible isolated vertices. Now, we prove that the diameter of the unique connected component of $\Gamma(G)$ is less than or equal to 3. If G has a unique maximal subgroup, then, by Lemma 2.1.1(1), G is a cyclic p-group and hence $\Gamma(G)$ is edgeless. So, we assume that G has at least two distinct maximal subgroups and we denote the set of all maximal subgroups of G by $\mathcal{M}$. Let H and K be two vertices in the component of $\Gamma(G)$. Then, H and K are not isolated vertices and by Theorem 2.2.1, $H, K \nsubseteq \Phi(G)$, i.e., there exists maximal subgroups M_1 and M_2 in $\mathcal{M}$ such that $H \nsubseteq M_1$ and $K \nsubseteq M_2$. If $M_1 = M_2$, then $M_1 \subsetneq HM_1 = G = KM_1 \supsetneq M_1$ (since G is nilpotent and M_1 is a maximal subgroup), i.e., we have a path $H \sim M_1 \sim K$ in $\Gamma(G)$ and $d(H, K) \leq 2$. Similarly, if $M_1 \neq M_2$, then we have a path $H \sim M_1 \sim M_2 \sim K$, i.e., $d(H, K) \leq 3$. Thus the diameter of the component of $\Gamma(G)$ is less than or equal to 3. □

Corollary 2.2.5 *Let G be a finite solvable group. Then $\Gamma(G)$ is connected apart from some possible isolated vertices and the component has diameter at most 4.*

Proof : It suffices to prove that a finite solvable group G always have a maximal subgroup which is normal in G. For finite groups, an equivalent definition of solvability is as follows: A finite group G is solvable if there are subgroups $\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{k-1} \trianglelefteq G_k = G$ such that each factors G_{i+1}/G_i is a cyclic group of prime order. So in particular, G/G_{k-1} is a cyclic group of prime order. Hence G_{k-1} is a maximal subgroup of G which is normal in G. □

Remark 2.2.6 *In Theorem 2.2.4, we have proved that if G has a maximal subgroup which is normal in G, then the diameter of the connected component is at most 4. However, we are yet to find an example of a group G, where the diameter of the component is equal to 4.*

Theorem 2.2.1 and Theorem 2.2.4, motivates us to put forward the next definition.

Definition 2.2.1 *Let G be a group. The deleted co-maximal subgroup graph of G, denoted by $\Gamma^*(G)$, is defined as the graph obtained by removing the* *isolated vertices from* $\Gamma(G)$*.*

Thus we have the immediate corollary:

Corollary 2.2.7 *Let* G *be a finite solvable group. Then* $\Gamma^*(G)$ *is connected and* $diam(\Gamma^*(G)) \leq 4$*. If* G *is nilpotent, then* $diam(\Gamma^*(G)) \leq 3$ *and* $\Gamma^*(G) = \Gamma(G)$ *if and only if* $\Phi(G)$ *is trivial.*

Remark 2.2.8 *There exists groups like* S_n *with* $n \geq 5$ *which are not solvable but has a maximal subgroup* A_n *which is normal in* S_n*. Thus there exists finite non-solvable groups* G *such that* $\Gamma^*(G)$ *is connected. Presently, authors are not aware of any finite non-solvable group* G *such that* $\Gamma^*(G)$ *is disconnected. Till now, no example of finite group* G *is known for which* $\Gamma^*(G)$ *is disconnected.*

Let $\Gamma_{max}(G)$ be the induced subgraph of $\Gamma(G)$ on the set of all maximal proper subgroups of G. If H is not an isolated vertex in $\Gamma(G)$, i.e., H is a vertex in $\Gamma^*(G)$, then any maximal subgroup of G containing H is at a distance 2 from H in $\Gamma(G)$. Thus each component of $\Gamma^*(G)$ must contain at least one maximal subgroup of G. Hence the number of components of $\Gamma^*(G)$ is bounded above by the number of maximal subgroups of G. Thus the number of components of $\Gamma^*(G)$ is bounded above by the number of components of $\Gamma_{max}(G)$. On the other hand, let C be a component of $\Gamma^*(G)$ and $M_1, M_2, \ldots, M_k$ be the maximal subgroups in C. Then $M_1, M_2, \ldots, M_k$ are also connected in $\Gamma_{max}(G)$. Thus the number of components of $\Gamma_{max}(G)$ is bounded above by the number of components of $\Gamma^*(G)$.

Thus we have the following theorem.

Theorem 2.2.5 *The number of components of* $\Gamma^*(G)$ *is equal to the number of components of* $\Gamma_{max}(G)$. *In particular,* $\Gamma_{max}(G)$ *is connected if and only if* $\Gamma^*(G)$ *is connected.*

2.3 Some Characterizations of $\Gamma(G)$ and $\Gamma^*(G)$

In this chapter, we characterize some graph properties like completeness, bipartiteness, girth etc. of $\Gamma(G)$ and $\Gamma^*(G)$. We note that authors in [6], (See Theorem 3.5 in [6] or Theorem 2.1.4) proved that if G is nilpotent, then $\Gamma(G)$ has an universal vertex if and only if $G \cong \mathbb{Z}_p \times \mathbb{Z}_q$. In fact, we characterize when $\Gamma(G)$ and $\Gamma^*(G)$ has an universal vertex, are complete, are star graph etc.

Theorem 2.3.1 $\Gamma(G)$ *is a complete graph on more than one vertices if and only if* $G \cong \mathbb{Z}_p \times \mathbb{Z}_q$, *where* p *and* q *are (not necessarily distinct) primes.*

Proof : We first assume that $G \cong \mathbb{Z}_p \times \mathbb{Z}_q$. If $p \neq q$, then $\Gamma(G) \cong \mathbb{Z}_2$ and hence complete. If $p = q$, then G has exactly $p+1$ subgroups of order p, say $H_1, H_2, \cdots, H_{p+1}$ and no other proper nontrivial subgroups. Hence $S = \{H_1, H_2, \cdots, H_{p+1}\}$. Note that $|H_iH_j| = \frac{|H_i||H_j|}{|H_i \cap H_j|} = p^2$. Thus $H_iH_j = G$ for $i \neq j$. Thus $\Gamma(G)$ is a complete graph of order $p+1$.

Conversely, we assume that $\Gamma(G) = K_n$ of order $n > 2$ and let $H_1, H_2, \cdots, H_n$ be the subgroups of G which form a complete graph. Clearly, $H_i \nsubseteq H_j$

and $H_i \cap H_j = \{e\}$ for all i, j with $i \neq j$, because if $H_i \subseteq H_j$, then $H_iH_j \subseteq H_j \neq G$, i.e H_i is not adjacent to H_j which contradicts by our assumption. Also, if $H_i \cap H_j = K \neq \{e\}$, then $H_iK \subseteq H_i \neq G$, i.e H_i is not adjacent to K which contradicts by our assumption. Similarly H_i's do not have any proper subgroup and hence H_i's are prime order subgroups of G of order p_i. Now as H_i is adjacent to H_j for all i, j with $i \neq j$, we have $H_iH_j = G$, i.e $|G| = |H_i||H_j| = p_ip_j$ for all i, j with $i \neq j$. Thus all p_i's equal, say $p_i = p$ for all i. Therefore, $|G| = p^2$. So $G = \mathbb{Z}_{p^2}$ or $G = \mathbb{Z}_p \times \mathbb{Z}_p$. Since $\Gamma(G)$ has no isolated vertex, G is not cyclic. Hence $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $n = p + 1$, so $p = n - 1$ is a prime. If $\Gamma(G) = K_2$, then it trivially follows that $G \cong \mathbb{Z}_p \times \mathbb{Z}_q$, for distinct primes p and q. □

Theorem 2.3.2 *Let G be a group of order n. $\Gamma(G)$ is a star graph $K_{1,p}$ if and only if G is a group of order $n = pq$, where $p > q$ are distinct primes.*

Proof : Let $\Gamma(G)$ be a star $K_{1,p}$, where H is the universal vertex and $K_1, K_2, \cdots, K_p$ are leaves [see Figure 2.3]. If H has any proper nontrivial subgroup L, then $HL \neq G$. Therefore H is not adjacent to L. However as H is a universal vertex, no such L exist. Thus H has no non-trivial proper subgroup, it means H is a subgroup of prime order, say η. Also, we have $HK_i = G$, for all $i = 1, 2, \cdots, p$. Thus $n = |G| = \frac{|H||K_i|}{|H \cap K_i|} = \eta|K_i|$. Hence, $|K_i| = \frac{n}{\eta}$, for all $i = 1, 2, \cdots, p$.

Note that as all K_i's are of the same order, they are not contained in each other. Moreover, K_i's can not have any non-trivial proper subgroups (otherwise if $L \subsetneq K_i$ is a subgroup, then L must be adjacent to H, i.e., $L =$

K_j. But that means $|K_i| > |L| = |K_j| = |K_i|$, a contradiction). Therefore, K_i's are prime order subgroups. Hence $\frac{n}{\eta}$ is prime, say q. So $n = q\eta$. Note that $q \neq \eta$, as otherwise $n = q^2$ and $G = \mathbb{Z}_{q^2}$ or $G = \mathbb{Z}_q \times \mathbb{Z}_q$. In none of the case, $\Gamma(G)$ is a star. Thus, G has p subgroups $K_1, K_2, \cdots, K_p$ each has order q. By Sylow's Theorem, the number of Sylow q-subgroups, n_q is given by $p = n_q = 1 + tq|\eta$. Hence, $p|\eta$. So $p = \eta$ and $n = pq$. Also, as K_1 is not adjacent to K_2. we have $K_1K_2 \neq G$. So, $pq = |G| > |K_1K_2| = \frac{|K_1||K_2|}{|K_1 \cap K_2|} = q^2$. It means $p > q$.

For the other direction, let G be a group of order pq, with $p > q$. Then $G \cong \mathbb{Z}_{pq}$ or $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ and $q|(p-1)$. In the first case, we get $\Gamma(G) \cong P_2$, which is a star. In the second case, it is easy to see that G has a normal subgroup H of order p and p subgroups $K_1, K_2, \cdots, K_p$ each has order q. Now, it is easy to check that $\Gamma(G)$ is a star with H as the universal vertex.

□

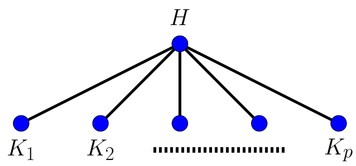


Figure 2.3: $\Gamma(G)$ is a star graph $K_{1,p}$

Remark 2.3.1 *From [6] Theorem 6.2 and Theorem 2.3.2, it follows that for a finite group G, $\Gamma(G)$ is a star if and only if $\Gamma(G)$ is a tree if and only if G is group of order pq, where p and q are distinct primes.*

Theorem 2.3.3 *Let G be a finite group. Then $\Gamma(G)$ has an universal vertex if and only if either G is non-cyclic abelian group of order p^2 or G is group of order pq, where p and q are distinct primes.*

Proof : Let H be a non-trivial proper subgroup of G such that H is an universal vertex in $\Gamma(G)$. Clearly H is both maximal and minimal subgroup in G, as otherwise H fails to be an universal vertex. Thus H is a prime order subgroup of G of order, say p. Thus $|G| = pm$. Clearly $m \neq 1$, i.e., $|G| \neq p$, as in that case $\Gamma(G)$ has no vertex.

If $p|m$, then H is contained in some Sylow p-subgroup K of G and hence $HK = K$. If $K \neq G$, then we get a contradiction as H is an universal vertex in $\Gamma(G)$. If $K = G$, then G is a p-group. Let $|G| = p^k$. If $k \geq 4$, then G has a subgroup L of order p^2 and it is easy to see that $|HL| \leq p^3$, i.e., $H \not\sim L$ in $\Gamma(G)$, a contradiction. Thus $|G| = p^2$ or p^3. If G is cyclic, then, by Theorem 2.2.2(1), $\Gamma(G)$ is edgeless, a contradiction. Thus G is either a non-cyclic abelian group of order p^2 or a non-cyclic group of p^3.

If G is a non-cyclic abelian group of order p^2, then $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and by Theorem 2.3.1, $\Gamma(G)$ is a complete graph. If G is a non-cyclic group of order p^3, then there are exactly two non-abelian groups and two abelian groups of order p^3, upto isomorphism. We discuss each of these possibilities separately:

Abelian Groups: In this case, $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ or $G \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2}$. In both cases, for every subgroup of order p, we can find another subgroup of order p whose product is not equal to G, i.e., $\Gamma(G)$ does not have any universal vertex.

Non-Abelian Groups: In this case, it is well known that the Frattini subgroup of a non-abelian group of order p^3 is non-trivial. Thus it always have isolated vertices, a contradiction.

Next we consider the case when $p \nmid m$. If m has two distinct prime factors q and r, then, by Cauchy's theorem, there exist a subgroup K of order q and $|HK| = pq < pm$. Even if $m = q^t$ where q is a prime and $t \geq 2$, we get a subgroup K of order q and $|HK| = pq < pm$. So only possibility is $m = q$, i.e., $|G| = pq$. Let $p > q$. If $q \nmid (p-1)$, then G is cyclic and $\Gamma(G) \cong K_2$. If $q \mid (p-1)$, then G is either cyclic or a non-abelian group of order pq. In the latter case, G has unique normal subgroup H of order p and all other vertices of $\Gamma(G)$ are adjacent to H.

Combining all the cases, it follows that if $\Gamma(G)$ has a universal vertex, then either G is non-cyclic abelian group of order p^2 or G is group of order pq, where p and q are distinct primes.

The converse follows from Theorem 2.3.1 and Theorem 2.3.2. □

Theorem 2.3.4 *Let G be a nilpotent group. $\Gamma^*(G)$ is a star if and only if either G is a group of order pq or the cyclic group of order $p^r q$, where p, q are distinct primes and $r \geq 2$ is an integer.*

Proof : If G is a cyclic group of order $p^r q$, then $G \cong \mathbb{Z}_{p^r q}$ and $\Gamma(G)$ is the union of isolated vertices $\langle pq \rangle, \langle p^2 q \rangle, \ldots, \langle p^{r-1} q \rangle$ and a star, where $\langle q \rangle$ is the universal vertex and $\langle p \rangle, \langle p^2 \rangle, \ldots, \langle p^{r-1} \rangle$ are the leaves. If G is a group of order pq, then by Theorem 2.3.2, $\Gamma(G) = \Gamma^*(G)$ is a star.

Conversely, let G be a nilpotent group such that $\Gamma^*(G)$ is a star. Then two cases may occur.

Case 1. $\Gamma(G) = \Gamma^*(G)$**:** Then by Theorem 2.3.2, G is a group of order $n = pq$.

Case 2. $\Gamma(G) \neq \Gamma^*(G)$**:** This means $\Gamma(G)$ has at least one isolated vertex. Let H be the universal vertex of the star $\Gamma^*(G)$ and $K_1, K_2, \ldots, K_t$ be the leaves of the star.

Claim 1: H is a maximal subgroup of G: If it is not, then there exists a subgroup L of G such that $H \subsetneq L \subsetneq G$. However, this impies $G = HK_i \subsetneq LK_i$, i.e., $LK_i = G$, i.e., $H \neq L$ and $L \sim K_i$ for $i = 1, 2, \ldots, t$, which contradicts that it is a star.

Claim 2: G is a cyclic group: If G has a unique maximal subgroup H, then by Lemma 2.1.1(1), G is a cyclic p-group, which implies that $\Gamma(G)$ is edgeless, a contradiction. Thus G has at least one maximal subgroup M, other than H. As G is nilpotent, both H and M, being maximal subgroups, are normal in G and $H \subsetneq HM = G$, i.e., $H \sim M$ in $\Gamma(G)$. However, this means $M = K_i$ for some $i \in \{1, 2, \ldots, t\}$. Now, choose $a \in G \setminus (H \cup M)$ (note that $G \neq H \cup M$) and set $A = \langle a \rangle$. If A is a proper subgroup of G, then $M \subsetneq MA = G$, i.e., $K_i = M \sim A$. Now, as K_i is a leaf, A must be the universal vertex H, i.e., $A = H$. However, as $a \notin H$, we have $A \neq H$, a contradiction. Hence, A is not a proper subgroup of G, i.e., $A = G = \langle a \rangle$ is cyclic.

Let $|G| = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$. If $k \geq 3$, then we get a clique of size k, namely $\langle p_1\rangle, \langle p_2\rangle, \ldots, \langle p_k\rangle$, a contradiction. Also $k = 1$ implies that $\Gamma(G)$ is edgeless. Thus $k = 2$, i.e., $|G| = {p_1}^{\alpha_1}{p_2}^{\alpha_2}$. If possible, let both $\alpha_1, \alpha_2 \geq 2$. Then, we get a 4-cycle, namely $\langle p_1\rangle, \langle p_2\rangle, \langle p_1^2\rangle, \langle p_2^2\rangle$, a contradiction. If both $\alpha_1 = \alpha_2 = 1$, then $\Gamma(G) \cong K_2$, i.e., without any isolated point. Thus, $|G|$ is of the required form. □

Theorem 2.3.5 *Let G be a nilpotent group. $\Gamma^*(G)$ is a complete graph if and only if either G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ or Q_8.*

Proof : If $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$ or Q_8, then the result follows from Theorem 2.3.1 and Figure 2.2(A).

Conversely, let $\Gamma^*(G)$ be a complete graph. If $\Gamma(G) = \Gamma^*(G)$, then by Theorem 2.3.1, $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p. So, we assume that $\Gamma(G) \neq \Gamma^*(G)$, i.e., $\Gamma(G)$ has at least one isolated vertex. Then by Theorem 2.2.1(2), the Frattini subgroup of G, $\Phi(G)$ is non-trivial. Let $H_1, H_2, \ldots, H_n$ be the vertices of $\Gamma^*(G)$.

Claim 1: Each H_i is a maximal subgroup of G.

Proof of Claim 1: Let there exists a proper subgroup L with $H_i \subsetneq L$. As $H_i \sim H_j$ in $\Gamma^*(G)$, we have $LH_j \supsetneq H_iH_j = G$, i.e. $L \sim H_j$, i.e., L is a vertex of $\Gamma^*(G)$. But $LH_i \subsetneq L \neq G$, which contradicts that $\Gamma^*(G)$ is complete. Hence H_i is a maximal subgroup of G.

Thus, all the maximal subgroups of G are vertices of $\Gamma^*(G)$ and there exists at least two maximal subgroups of G. If the number of maximal subgroups is exactly 2, then by Proposition 2.1.1(2), $G \cong \mathbb{Z}_{p^a q^b}$. Now as $\Gamma(G)$ has at least one isolated vertex, we have $ab > 1$, i.e., either $a > 1$ or

$b > 1$. Suppose $a > 1$. Then $\langle p^2 \rangle$ is not an isolated vertex and $\langle p \rangle \not\sim \langle p^2 \rangle$ in $\Gamma(G)$. This contradicts that $\Gamma^*(G)$ is complete. Thus the number of maximal subgroups of G is at least 3.

Claim 2: Each H_i is cyclic.

Proof of Claim 2: Let $a \in H_i \setminus \Phi(G)$ and $A = \langle a \rangle$. If $A \subsetneq H_i$, then we get a proper subgroup of G which is not contained in $\Phi(G)$. Thus A is a vertex of $\Gamma^*(G)$. But $AH_i \subseteq H_i \neq G$, i.e., $A \not\sim H_i$, which contradicts that $\Gamma^*(G)$ is complete. Thus $A = H_i = \langle a \rangle$, i.e., H_i is cyclic.

As all the maximal subgroups of G are cyclic, it follows that all subgroups of G are cyclic. Next we prove that all non-maximal subgroups are contained in $\Phi(G)$.

Claim 3: Any proper subgroup of H_i is contained in $\Phi(G)$.

Proof of Claim 3: If possible, let there exists a subgroup L of G such that $\Phi(G) \subsetneq L \subsetneq H_i$. Then L is a vertex of $\Gamma^*(G)$ and $LH_i \subseteq H_i \neq G$, i.e., $L \not\sim H_i$, a contradiction, as $\Gamma^*(G)$ is complete.

As $\Phi(G)$ is normal in G, it is also normal in each H_i. Now, it follows from the above claim that $H_i/\Phi(G)$ has no non-trivial proper subgroup, i.e., $|H_i/\Phi(G)| = p_i$, for some prime p_i, for $i = 1, 2, \ldots, n$.

Now, as $H_i \sim H_j$ in $\Gamma^*(G)$, we have $H_iH_j = G$, i.e.,

$$|G| = |H_iH_j| = \frac{|H_i||H_j|}{|H_i \cap H_j|} = \frac{p_i|\Phi(G)| \cdot p_j|\Phi(G)|}{|\Phi(G)|} = p_ip_j|\Phi(G)|.$$

Since G has at least 3 maximal subgroups, we have $p_i = p_j = p$(say) for all $i, j \in \{1, 2, \ldots, n\}$. Hence $|G| = p^2|\Phi(G)|$ and $|H_i| = p|\Phi(G)|$, i.e.,

$|G/\Phi(G)| = p^2$. Hence, by Proposition 2.1.1(4), G is a p-group. Let $|G| = p^k$. Then $|H_i| = p^{k-1}$.

Now, we try to classify the group G. If G is abelian, then as G has cyclic subgroups of order p^{k-1}, we have $G \cong \mathbb{Z}_{p^k}$ or $\mathbb{Z}_{p^{k-1}} \times \mathbb{Z}_p$. The former can not hold as in that case $\Gamma(G)$ is edgeless. Thus $G \cong \mathbb{Z}_{p^{k-1}} \times \mathbb{Z}_p$. However, $\langle p \rangle \times \mathbb{Z}_p$, being a subgroup of order p^{k-1} of $\mathbb{Z}_{p^{k-1}} \times \mathbb{Z}_p$, is a maximal subgroup which is not cyclic, a contradiction. Thus $G \ncong \mathbb{Z}_{p^{k-1}} \times \mathbb{Z}_p$.

Hence G is a non-abelian group of order p^k. Then G is a minimal non-cyclic p-group. Thus, by Theorem 2.1.1, G is isomorphic to Q_8. (As G is a non-abelian p-group, other two cases do not occur.) $\square$

Theorem 2.3.6 *Let G be a finite nilpotent group. $\Gamma^*(G)$ has an universal vertex if G is isomorphic to one of the following groups:*

1. $\mathbb{Z}_{p^r q}$, *where p, q are distinct primes.*

2. $\mathbb{Z}_p \rtimes \mathbb{Z}_q$, *where p, q are distinct primes and $q | p - 1$.*

3. $\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p$, *where p is a prime.*

4. $M_{p^n} = \langle a, b : a^{p^{n-1}} = b^p = e; b^{-1}ab = a^{1+p^{n-2}} \rangle$, *where p is a prime.*

5. $D_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = e; bab = a^{-1} \rangle$.

6. $Q_{2^n} = \langle a, b : a^{2^{n-1}} = e; b^2 = a^{2^{n-2}}; b^{-1}ab = a^{-1} \rangle$.

7. $SD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = e; bab = a^{-1+2^{n-2}} \rangle$.

Proof : Let $\Gamma^*(G)$ has an universal vertex. If $\Gamma^*(G) = \Gamma(G)$, then G is as described in Theorem 2.3.3. If $\Gamma^*(G) \neq \Gamma(G)$, then $\Phi(G)$ is non-trivial. Let H be an universal vertex of $\Gamma^*(G)$.

Claim 1: H is a maximal subgroup.

Proof of Claim 1: If not, then there exists a proper subgroup L with $H \subsetneq L \neq G$. Then $HL = L \neq G$, i.e., contradicting that H is an universal vertex, unless L is an isolated vertex of $\Gamma(G)$. However, in that case, $H \subsetneq L \subseteq \Phi(G)$, a contradiction.

Claim 2: H is cyclic.

Proof of Claim 2: If G has exactly one maximal subgroup H, then G is a cyclic p-group and hence $\Gamma^*(G)$ is an empty graph. Thus G has at least two maximal subgroups and hence $\Phi(G) \subsetneq H$. Let $a \in H \setminus \Phi(G)$. Then $\langle a \rangle \subseteq H$ and $\langle a \rangle \not\subseteq \Phi(G)$. If $\langle a \rangle$ is a proper subgroup of H, then $\langle a \rangle H = H \neq G$, i.e., $\langle a \rangle \nsim H$ in $\Gamma^*(G)$, a contradiction. Thus $H = \langle a \rangle$.

Note that all subgroups of H are contained in $\Phi(G)$, i.e., $H/\Phi(G)$ has no non-trivial proper subgroups. This implies that $H/\Phi(G)$ is a cyclic group of prime order, say p. Then $|H| = p|\Phi(G)|$.

Also note that all elements in $H \setminus \Phi(G)$ are generators of the cyclic group H. Thus by equating the number of generators of a finite cyclic group, we get

$$|H| - |\Phi(G)| = \varphi(|H|), \text{ i.e., } (p-1)|\Phi(G)| = \varphi(|H|), \qquad (2.1)$$

where φ denote the Euler's totient function.

Claim 3: Intersection of any maximal subgroup of G with H is $\Phi(G)$.

Proof of Claim 3: Let K be a maximal subgroup of G other than H. Then $H \cap K \subsetneq H$. As proper subgroups of H are contained in $\Phi(G)$, we have $H \cap K \subseteq \Phi(G)$. On the other hand, $\Phi(G)$ being the intersection of all maximal subgroups is contained in $H \cap K$. Thus $H \cap K = \Phi(G)$.

Thus, the intersection number of G, $\iota(G) = 2$. As G is nilpotent, by Theorem 4.7 of [41], we have

$$2 = \iota(G) = \sum_{i=1}^{k} \iota(P_i) = \sum_{i=1}^{k} r_i,$$

where $\pi(G) = \{p_1, p_2, \ldots, p_k\}$ and P_i is a Sylow p_i-subgroup of G with rank r_i. As the only partitions of 2 are $1+1$ and $2+0$, $|\pi(G)| \leq 2$.

Case 1: $2 = 1 + 1$. In this case, $\pi(G) = \{p, q\}$ and $|G| = p^a q^b$ and $G \cong P \times Q$, where P, Q are Sylow p and q subgroups of G respectively of rank 1 each, i.e., $|P/\Phi(P)| = p$ and $|Q/\Phi(Q)| = q$. Thus

$$|G/\Phi(G)| = \frac{|G|}{|\Phi(G)|} = \frac{|P \times Q|}{|\Phi(P) \times \Phi(Q)|} = |P/\Phi(P)||Q/\Phi(Q)| = pq,$$

i.e., $|\Phi(G)| = p^{a-1}q^{b-1}$ and $|H| = p^a q^{b-1}$. Thus, if $b > 1$, from Equation 2.1, we get

$$(p-1)p^{a-1}q^{b-1} = \varphi(p^a q^{b-1}) \Rightarrow q = q - 1, \text{ a contradiction.}$$

Thus, the only possibility is $b = 1$, i.e., $|G| = p^a q, |H| = p^a, |\Phi(G)| = p^{a-1}$. Moreover, as $\Phi(G)$ is non-trivial, we have $a \geq 2$. Also, as H is a maximal subgroup of a nilpotent group G, H is normal in G. Hence, $H = P$ is the unique Sylow p-subgroup of G and $G \cong H \times Q$, where Q is the Sylow q-subgroup of G of order q. Thus G is a cyclic group of order $p^a q$, i.e., $G \cong \mathbb{Z}_{p^a q}$, with $a \geq 2$.

Case 2: $2 = 2 + 0$**.** In this case, G is a p-group, i.e., say $|G| = p^n$ and $|H| = p^{n-1}, \Phi(G) = p^{n-2}$. Moreover G can not be cyclic, as in that case $\Gamma^*(G)$ will be null graph. Thus, G is a non-cyclic p-group with a cyclic subgroup of index p. Then by Theorem 1.2 [16], G is isomorphic to one of the following groups:

- $\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p$
- $M_{p^n} = \langle a, b : a^{p^{n-1}} = b^p = e; b^{-1}ab = a^{1+p^{n-2}} \rangle$
- $D_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = e; bab = a^{-1} \rangle$
- $Q_{2^n} = \langle a, b : a^{2^{n-1}} = e; b^2 = a^{2^{n-2}}; b^{-1}ab = a^{-1} \rangle$
- $SD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = e; bab = a^{-1+2^{n-2}} \rangle$.

The converse part follows immediately from the following observations:

- $\mathbb{Z}_{p^{n-1}} \times \{[0]\}$ is an universal vertex in $\Gamma^*(G)$, when $G \cong \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p$.
- $\langle a \rangle$ is an universal vertex in $\Gamma^*(G)$, when $G \cong M_{p^n}, D_{2^{n-1}}, Q_{2^n}$ or SD_{2^n}.

□

Corollary 2.3.2 *Let G be a finite nilpotent group. The domination number of $\Gamma^*(G)$ is 1 if and only if G is one of the groups mentioned in Theorem 2.3.6.* □

Theorem 2.3.7 *Let G be a finite nilpotent group. Then $\Gamma(G)$ is bipartite if and only if G is a cyclic group of order p^a or p^aq^b, where p, q are distinct primes.*

Proof : By Theorem 2.2.4, it follows that $\Gamma(G)$ is connected, except a few possible isolated vertices. As isolated vertices does not affect the bipartiteness of a graph, we ignore the isolated vertices. If G has a unique maximal subgroup, then G is cyclic p-group and $\Gamma(G)$ is edgeless and hence bipartite.

If G has at least 3 maximal subgroups, say M_1, M_2 and M_3, then, as G is nilpotent, we have $M_1M_2 = M_2M_3 = M_3M_1 = G$, i.e., we get a 3-cycle $M_1 \sim M_2 \sim M_3 \sim M_1$. Hence $\Gamma(G)$ is non-bipartite.

So we assume that G has exactly two maximal subgroups M_1 and M_2. Then, by Proposition 2.1.1(2), G is a cyclic group of p^aq^b, where p, q are distinct primes, i.e., $G \cong \mathbb{Z}_{p^aq^b}$. Then the vertices of $\Gamma^*(G)$ can be partitioned into two partite sets $V_1 = \{\langle p\rangle, \langle p^2\rangle, \ldots, \langle p^a\rangle\}$ and $V_2 = \{\langle q\rangle, \langle q^2\rangle, \ldots, \langle q^b\rangle\}$, thereby making it bipartite. $\square$

Theorem 2.3.8 *Let G be a finite nilpotent group. Then*

$$girth(\Gamma^*(G)) = \begin{cases} \infty, & \textit{if } G \cong \mathbb{Z}_{p^a} \textit{ or } \mathbb{Z}_{p^aq}, \textit{ where } a \geq 1 \\ 4, & \textit{if } G \cong \mathbb{Z}_{p^aq^b}, \textit{ where } a, b \geq 2 \qquad p, q \textit{ are distinct primes.} \\ 3, & \textit{otherwise.} \end{cases}$$

.

Proof : By Corollary 2.2.7, it follows that $\Gamma^*(G)$ is connected. If G has a unique maximal subgroup, then G is cyclic p-group and $\Gamma(G)$ is edgeless.

If G has at least three maximal subgroups, say M_1, M_2 and M_3, then, as G is nilpotent, we have $M_1M_2 = M_2M_3 = M_3M_1 = G$, i.e., we get a 3-cycle $M_1 \sim M_2 \sim M_3 \sim M_1$. Hence girth of $\Gamma^*(G)$ is 3.

So we assume that G has exactly two maximal subgroups M_1 and M_2. Then by Proposition 2.1.1(2), $G \cong \mathbb{Z}_{p^a q^b}$ and by Theorem 2.3.7, $\Gamma^*(G)$ is bipartite. If any of a or b is 1, then $\Gamma(G)$ is a star and has no cycle, i.e., girth of $\Gamma(G)$ is ∞. If both $a, b \geq 2$, then we have the 4-cycle $\langle p \rangle \sim \langle q \rangle \sim \langle p^2 \rangle \sim \langle q^2 \rangle \sim \langle p \rangle$ in $\Gamma^*(G)$ and hence girth of $\Gamma^*(G)$ is 4. $\square$

2.4 Independence Number and Chromatic Number of $\Gamma(G)$

In this section, we study the independence number α and chromatic number χ of $\Gamma(G)$ and prove that $\Gamma(G)$ is weakly perfect.

Lemma 2.4.1 *Let G be a finite group with at most 7 non-trivial proper subgroups. Then G is supersolvable.*

Proof : If G is nilpotent, then G is supersolvable. Thus we assume that G is not nilpotent. As every finite group whose all Sylow subgroups are cyclic is supersolvable (Theorem 10.1.10 [36]), it suffices to show that all Sylow subgroups of G are cyclic. Moreover, as G is not nilpotent, G is not a p-group, i.e., $\pi(G) \geq 2$ i.e., there exists two distinct primes p and q dividing $|G|$. Let P be a Sylow p-subgroup and Q be a Sylow q-subgroup of G. If possible, let P be a non-cyclic group. Then P has at least 3 non-trivial proper subgroups, say P_1, P_2, P_3.

Case 1: $(n_p(G), n_q(G) > 1)$ If $|\pi(G)| = 2$, then $n_p(G) + n_q(G) \geq 5$. (If $|\pi(G)| = 2$, then both $n_p(G), n_q(G)$ can not be simultaneously equal to 2.) Also P has 3 non-trivial proper subgroups, thereby making the number of proper subgroups of G to 8, a contradiction. If If $|\pi(G)| \geq 3$, then $n_p(G) + n_q(G) \geq 4$ and let r be another element of $\pi(G)$ other than p and q. Then Sylow r-subgroup and three subgroups of P, take the total count to at least 8, a contradiction.

Case 2: $(n_p(G) = 1, n_q(G) > 1)$ In this case, P is a normal subgroup of G and $H = PQ$ is a subgroup of G. As $n_q(H)$ divides $|P|$, we have $n_q(H) = 1$ or $n_q(H) = p^s$ for some positive integer s. If $n_q(H) = 1$, then Q is normal in H and $P, Q, P_1, P_2, P_3, P_1Q, P_2Q, P_3Q$ are 8 distinct subgroups of H, a contradiction. If $n_q(H) = p$, then $p = 1 + ql$ for some integer l, i.e., $q|p-1$ and so $p > q \geq 2$, i.e, $p \geq 3$. Now as $n_q(G) > 1$, therefore $n_q(G) = 1 + qt \geq 3$, i.e., G has at least 3 Sylow q-subgroups. Also P, being a non-cyclic p-group has at least $p + 1$ non-trivial proper subgroups. Thus G has at least 8 proper non-trivial subgroups, namely P, at least $p + 1 \geq 4$ non-trvial proper subgroups of P and at least 3 Sylow q-subgroups of G, a contradiction.

Case 3: $(n_p(G) > 1, n_q(G) = 1)$ Proof is similar to Case 2.

Case 4: $(n_p(G) = n_q(G) = 1)$ In this case, both P and Q are normal in G. Then $P, Q, P_1, P_2, P_3, P_1Q, P_2Q, P_3Q$ are 8 distinct subgroups of G, a contradiction.

Combining all the cases, we see that all the Sylow subgroups are cyclic and hence G is supersolvable. □

Theorem 2.4.1 *Let G be a finite group such that $\alpha(\Gamma(G)) \leq 8$. Then G is solvable.*

Proof : As $\alpha(\Gamma(G)) \leq 8$, every maximal subgroup M of G contains at most 7 proper non-trivial subgroups (otherwise M along with its proper non-trivial subgroups forms an independent set of size more than 8). Thus, by Lemma 2.4.1, M is supersolvable. Thus every maximal subgroup of G is supersolvable and hence, by Theorem 10.3.4 [36], G is solvable. □

Remark 2.4.2 *Thus for every finite non-solvable group G, $\Gamma(G)$ has independence number at least 9.*

It was proved in Theorem 6.4, [6] that if G be a finitely generated nilpotent group, then the clique number $\omega(\Gamma(G)) = \chi(\Gamma(G)) = |Max(G)|$, i.e., $\Gamma(G)$ is *weakly perfect*. The authors in [6] also raised the question that whether $\Gamma(G)$ is weakly perfect for all finite groups G. We answer this question assertively.

Theorem 2.4.2 *Let G be a finite group. Then $\Gamma(G)$ is weakly perfect.*

Proof : For any finite graph, the chromatic number is at least as large as the clique number. Thus it suffices to show that $\chi(\Gamma(G)) \leq \omega(\Gamma(G))$. Let $S = \{H_1, H_2, \ldots, H_\omega\}$ be a maximum clique of $\Gamma(G)$. As no two of them are simultaneously contained in any maximal subgroup, without loss of generality, we can assume each H_i to be maximal subgroups. Thus $|Max(G)| \geq \omega$.

We now assign the colour 1 to H_1 and all proper subgroups of H_1. Then we assign the colour 2 to H_2 and all proper subgroups of H_2 which are not contained in H_1. Proceeding this way, for each $i = 1, 2, \ldots, \omega$, we assign the colour i to H_i and all proper subgroups of H_i which are not contained in H_j, for $j < i$.

Note that $Max(G)$ may contain maximal subgroups other than those in S. Let $M \in Max(G) \setminus S$. Thus M is not adjacent to some H_i. Then we assign the colour i to M_i and all of its subgroups which have not been coloured previously. We repeat this process until all maximal subgroups are coloured. It is easy to see that all subgroups receive some colour and it forms a proper colouring of $\Gamma(G)$. Thus $\chi(\Gamma(G)) \leq \omega(\Gamma(G))$. Hence $\Gamma(G)$ is weakly perfect. □

2.5 Perfectness of $\Gamma(G)$

The authors in [6] mentioned that by using similar arguments to that of weak perfectness, it can be proved that $\Gamma(G)$ is perfect. However, note that $\Gamma(G)$ may not be perfect, even if G is a finite cyclic group. In fact, it has been proved in the next chapter that $\Gamma(\mathbb{Z}_n)$ is perfect if and only if n has atmost 4 distinct prime factors. In this section, we show that certain families of groups yield perfect graphs. Before that we recall the definition of perfectness of a graph. A graph $\mathcal{G}$ is called *perfect* if for all induced subgraphs H of $\mathcal{G}$, $\omega(H) = \chi(H)$. The *strong perfect graph theorem* states that a graph $\mathcal{G}$ is perfect if and only if $\mathcal{G}$ and its complement does not have induced odd cycle of length greater than or equal to 5. This result will be extensively used in this section.

Lemma 2.5.1 *Let G be a finite group and H be a subgroup with normal complement in G. If $\Gamma(G)$ is perfect, then $\Gamma(H)$ is perfect.*

Proof : Let K be a normal complement of H in G, i.e., $K \lhd G, HK = G$ and $H \cap K$ is trivial. Suppose $\Gamma(H)$ is not perfect. Then either $\Gamma(H)$ or $\Gamma^c(H)$ has an induced odd cycle of length ≥ 5. Suppose $\Gamma(H)$ has an induced odd cycle C of length $2t+1$, namely $H_1 \sim H_2 \sim \cdots H_{2t+1} \sim H_1$, where H_i's are non-trivial proper subgroups of H. Then $H_1K \sim H_2K \sim \cdots H_{2t+1}K \sim H_1K$ is an induced odd cyle in $\Gamma(G)$, a contradiction. (Note that the adjacency follows from normality of K in G and all H_iK's are distinct proper subgroups of G.) Similarly, it can be shown that $\Gamma^c(H)$ has no induced odd cycle of length ≥ 5. Hence $\Gamma(H)$ is perfect. □

Theorem 2.5.1 *If G is a group of order p^2q^2, then $\Gamma(G)$ is perfect*

Proof : We first consider the case when $|G| = 36 = 2^2 \cdot 3^2$. There are 15 non-isomorphic groups of order 36 and each of them has been checked in Sagemath to yield perfect comaximal subgroup graph. So, we focus on the case when $|G| \neq 36$. Without loss of generality, let $p > q$. We first show that $\Gamma(G)$ has no induced odd cycle of length ≥ 5. Suppose, $\Gamma(G)$ has an induced odd cycle $C : H_1 \sim H_2 \sim \cdots H_t$ with $t \geq 5$.

Claim 1: C has no subgroup of order q.

Proof of Claim: Let $|H_1| = q$. Then, as $H_t \sim H_1 \sim H_2$, we have

$|H_2| = |H_t| = p^2q$. However, $|H_2H_t| \geq p^4q^2/p^2 = p^2q^2 = |G|$, i.e., $H_2 \sim H_t$, a contradiction, as C is chordless.

The above argument also shows that C can have at most two subgroups of order p^2q, in which case they must be adjacent. However, we make a stronger claim.

Claim 2: C has no subgroup of order p^2q.

Proof of Claim: Let $|H_1| = p^2q$. Now, $H_3, H_4 \nsim H_1$ and $H_3 \sim H_4$. As $H_3, H_4 \nsim H_1$, $|H_3|, |H_4|$ can not be pq^2 or q^2. Thus $|H_3|, |H_4| \in \{p, pq, p^2\}$. As $H_3 \sim H_4$, we must have $|H_3| = |H_4| = pq$ and $|H_3 \cap H_4| = 1$. Again as $H_3, H_4 \nsim H_1$, we have $H_3, H_4 \subseteq H_1$. But this implies that $H_3H_4 \subseteq H_1 \neq G$, i.e., $H_3 \nsim H_4$, a contradiction. Thus C has no subgroup of order p^2q.

Claim 3: C has no subgroup of order q^2.

Proof of Claim: Let $|H_1| = q^2$. In light of Claim 2, we must have $|H_2| = |H_t| = p^2$. However, unless $|G| = 36$, any group of order p^2q^2 with $p > q$ has a unique Sylow p subgroup of order p^2. Thus $H_2 = H_t$, a contradiction.

Claim 4: C has no subgroup of order pq^2.

Proof of Claim: Let $|H_1| = pq^2$. Unless $p = 3$ and $q = 2$, i.e., $|G| = 36$, H_1 contains a unique subgroup of order p. Now, as $H_3, H_4 \nsim H_1$ and $H_3 \sim H_4$, both $|H_3|, |H_4|$ can not be p or p^2. If $|H_3| = p$, then $|H_4| = pq^2$ and $|H_3 \cap H_4| = 1$. Again, as $H_1 \nsim H_3$, we have $H_3 \subseteq H_1$, i.e., H_3 is the unique subgroup of order p in H_1. Also, as $H_1 \nsim H_4$, comparing their orders, we must have $|H_1 \cap H_4| = p$ or pq. In any case, H_4 contains a subgroup of order p, i.e, H_3. This contradicts that $|H_3 \cap H_4| = 1$. Thus C does not contain any subgroup of order p. Similarly it can be shown that C does not contain any subgroup of order p^2. Thus $|H_3|, |H_4| \in \{pq, pq^2\}$.

This gives rise to four cases, namely $(|H_3|, |H_4|) = (pq, pq), (pq, pq^2), (pq^2, pq)$ and (pq^2, pq^2). Using the fact that H_1 contains a unique subgroup of order p, in all the above cases, we get a contradiction. Hence C has no subgroup of order pq^2. Hence Claim 4 holds.

Using all the above claims, it also follows that C does not contain any subgroup of order p or p^2. Thus all the vertices in C must be subgroups of order pq.

Claim 5: No subgroup in C is cyclic.

Proof of Claim: If possible, let H_1 be a cyclic group of order pq. So, it has a unique subgroup of order p and a unique subgroup of order q. As $H_1 \nsim H_3$, we must have $|H_1 \cap H_3| = p$ or q. Suppose $|H_1 \cap H_3| = p$. Similarly, $H_1 \cap H_4$ is non-trivial. However, if $|H_1 \cap H_4| = p$, then both H_3, H_4 contains that unique subgroup of order p. But thus implies $H_3 \nsim H_4$. Hence, $|H_1 \cap H_4| = q$. Proceeding in this manner, we get $|H_1 \cap H_5| = p, |H_1 \cap H_6| = q, \ldots, |H_1 \cap H_t| = p$. But this implies $H_1 \nsim H_t$. Hence, Claim 5 holds.

Thus all the vertices in C must be non-abelian subgroups of order pq and hence $q|(p-1)$, i.e., each H_i is generated by two elements, one of order p and one of order q. Also note that adjacent vertices in C intersect trivially and non-adjacent vertices in C intersect in a subgroup of order p or q.

Claim 6: Any two adjacent vertices in C can not both intersect in a subgroup of order p with any other vertex of C.

Proof of Claim: Let $|H_1| = |H_3| = |H_4| = pq$ with $|H_1 \cap H_3| = |H_1 \cap H_4| = p$ and $|H_3 \cap H_4| = 1$. As $p > q$, H_1 contains a unique subgroup K of order p and hence $K \subseteq H_3, H_4$, contradicting that $H_3 \sim H_4$.

If the unique Sylow p subgroup P of G is cyclic, then G has a unique subgroup, say K, of order p. Thus K is contained in all subgroups in C, which contradicts that $H_1 \sim H_2$ in C. So, we assume that the unique Sylow p subgroup P of G is not cyclic, i.e., $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Again, if G is nilpotent, then all vertices in C are nilpotent groups of order pq, i.e., all the vertices in C are cyclic subgroups of G, contradicting Claim 5. Thus G is a non-nilpotent group with a unique Sylow p subgroup P, with $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Claim 7: Any two adjacent vertices in C can not both intersect in a subgroup of order q with any other vertex of C which is not adjacent to both.

Proof of Claim: Let $H_1 \not\sim H_m, H_{m+1}$ and, if possible, $|H_1 \cap H_m| = |H_1 \cap H_{m+1}| = q$. Also, we have $|H_1 \cap H_2| = |H_m \cap H_{m+1}| = 1$. Let $H_1 = \langle a_1, b_1 \rangle$ and $H_2 = \langle a_2, b_2 \rangle$ where $\circ(a_i) = p, \circ(b_i) = q$ for $i = 1, 2$ and $|\langle a_1 \rangle \cap \langle a_2 \rangle| = 1 = |\langle b_1 \rangle \cap \langle b_2 \rangle|$ and $H_1 H_2 = G = \langle a_1, a_2, b_1, b_2 \rangle$. Thus any element of G can be expressed uniquely in the form $a_1^i a_2^j b_1^k b_2^l$ where $0 \leq i, j \leq p-1$ and $0 \leq k, l \leq q-1$.

As $|H_1 \cap H_m| = q$, without loss of generality, we can assume $H_m = \langle a_3, b_1 \rangle$, where $\circ(a_3) = p$ and $|\langle a_1 \rangle \cap \langle a_3 \rangle| = 1$. Similarly, as $|H_1 \cap H_{m+1}| = q$, we can assume $H_{m+1} = \langle a_4, a_1^i b_1^j \rangle$, where $\circ(a_4) = p, \circ(a_1^i b_1^j) = q$ and $|\langle a_3 \rangle \cap \langle a_4 \rangle| =$

$|\langle a_1\rangle \cap \langle a_4\rangle| = 1$. Now $G = H_m H_{m+1} = \langle a_3, a_4, b_1, a_1^i b_1^j\rangle = \langle a_3, a_4, b_1, a_1^i\rangle$. Note that as $|\langle a_3\rangle \cap \langle a_4\rangle| = 1$, we have $\langle a_3, a_4\rangle = P$, the unique Sylow p subgroup of G. Thus a_1, being an element of order p, belongs to $\langle a_3, a_4\rangle$. Hence $G = \langle a_3, a_4, b_1\rangle$, i.e., G is generated by two elements of order p and an element of order q. This contradicts that $|G| = p^2q^2$. Hence Claim 7 holds.

By Claim 7, there exists a pair of non-adjacent vertices in C which intersect in a subgroup of order p. Without loss of generality, let $|H_1 \cap H_3| = p$. Then. by Claim 6, we must $|H_1 \cap H_4| = q$. Again, by Claim 7, we get $|H_1 \cap H_5| = p$. Continuing in this way, we get $|H_1 \cap H_t| = p$. But this contradicts that $H_1 \sim H_t$. Thus $\Gamma(G)$ does not have an induced odd cycle $C : H_1 \sim H_2 \sim \cdots H_t$ with $t \geq 5$. Similarly, it can be proved that $\Gamma^c(G)$ does not have an induced odd cycle of length ≥ 5. Hence the theorem holds. $\square$

Theorem 2.5.2 *If G is a group of order p^2qr with $p > q > r$, then $\Gamma(G)$ is perfect.*

Proof : We first show that $\Gamma(G)$ has no induced odd cycle of length ≥ 5. Suppose, $\Gamma(G)$ has an induced odd cycle $C : H_1 \sim H_2 \sim \cdots H_t$ with $t \geq 5$. As subgroups of G are of orders $p, p^2, q, r, pq, pr, qr, p^2q, p^2r, pqr$, we show that none of them can lie on C.

Claim 1: C has no subgroup of order p^2q.

Proof of Claim: Let $|H_1| = p^2q$. Then, H_1 is a maximal subgroup of G such that $H_1 \lhd G$. As $H_3, H_4 \not\sim H_1$, we must have $H_3, H_4 \subset H_1$. This implies that $H_3 \not\sim H_4$, a contradiction. Thus C has no subgroup of order p^2q. As subgroups of order r can only be adjacent to subgroups of order p^2q, C also has no subgroup of order r.

Claim 2: C has no subgroup of order p^2r.

Proof of Claim: Let $|H_1| = p^2r$. As $H_3, H_4 \not\sim H_1$, q does not divide $|H_3|$ and $|H_4|$, i.e., q does not divide $|H_3H_4|$. This impplies that $H_3 \not\sim H_4$, a contradiction. Thus C has no subgroup of order p^2r. As subgroups of order q can only be adjacent to subgroups of order p^2r, C also has no subgroup of order q.

Claim 3: C has no subgroup of order p^2.

Proof of Claim: Let $|H_1| = p^2$. As $H_2, H_t \sim H_1$, we must have $|H_2|, |H_t| \in \{qr, pqr\}$. Also, as $H_3, H_4 \not\sim H_1$, qr does not divide $|H_3|$ and $|H_4|$. Thus $|H_3|, |H_4| \in \{p, p^2, pq, pr\}$. Again, as $H_3 \sim H_4$, we get $|H_3|, |H_4| \neq p, p^2$, and hence without loss of generality, $|H_3| = pq, |H_4| = pr, \cdots, |H_{t-1}| = pr$.

As $H_2 \sim H_3$ and $H_{t-1} \sim H_t$, we get $|H_2| = |H_t| = pqr$ and $|H_1 \cap H_2| = |H_1 \cap H_t| = p$. Again, since $H_3 \not\sim H_t$, we have $|H_3 \cap H_t| = p$. As $p > q > r$, H_2, H_t, being subgroups of order pqr, has a unique subgroup P of order p, i.e., $P \subseteq H_1, H_2, H_3, H_t$. In particular, $P \subseteq H_2 \cap H_3$, i.e, p divides $|H_2 \cap H_3|$. On the other hand, as $H_2 \sim H_3$, we must have $|H_2 \cap H_3| = q$,a contradiction. Thus Claim 3 holds.

As subgroups of order qr can be adjacent only to subgroups of order p^2, p^2q, p^2r, none of which are in C, C does not have any subgroup of order qr.

Claim 4: C has no subgroup of order pqr.

Proof of Claim: Let $|H_1| = pqr$. Then $|H_2|, |H_t| \in \{p, pq, pr, pqr\}$. Note that in any case, p divides $|H_1 \cap H_3|$ and $|H_1 \cap H_4|$. As H_1 has a unique Sylow p subgroup P, $P \subseteq H_3, H_4$, i.e., p divides $|H_3 \cap H_4|$. This contradicts that $H_3 \sim H_4$. Thus Claim 4 holds. As subgroups of order p can be adjacent only to subgroups of order pqr, C does not have any subgroup of order p.

Thus C can have subgroups of order pq and pr, and they must alternate in the cycle for adjacency. However as C is an odd cycle, this can not occur. Thus $\Gamma(G)$ has no induced odd cycle of length ≥ 5. Similarly, it can be shown that $\Gamma^c(G)$ has no induced odd cycle of length ≥ 5. Thus $\Gamma(G)$ is perfect. □

Theorem 2.5.3 *If G is a group of order $pqrs$, then $\Gamma(G)$ is perfect.*

Proof : We first show that $\Gamma(G)$ has no induced odd cycle of length ≥ 5. Suppose, $\Gamma(G)$ has an induced odd cycle $C : H_1 \sim H_2 \sim \cdots H_t$ with $t \geq 5$.

Claim 1: C has no subgroups of order pqr, pqs, prs, qrs.

Proof of Claim: Let $|H_1| = pqr$. As $H_3, H_4 \not\sim H_1$, s does not divide $|H_3|, |H_4|$. Hence $H_3 \not\sim H_4$, a contradiction. Similarly it can be shown that C has no subgroups of order pqs, prs, qrs.

From Claim 1, it follows that C has no subgroups of order p, q, r or s. Thus only possible orders of subgroups in C are pq, pr, ps, qr, qs, rs. Suppose $|H_1| = pq$. Then, as $H_1 \sim H_2$, we must have $|H_2| = rs$. Similarly, $|H_3| = pq, |H_4| = rs$ and so on, i.e., subgroups of order pq and rs must alternate in C. But this contradicts that C is an odd cycle. Thus $\Gamma(G)$ has no induced odd cycle of length ≥ 5. Similarly, it can be shown that $\Gamma^c(G)$ has no induced odd cycle of length ≥ 5. Thus $\Gamma(G)$ is perfect. $\square$

Theorem 2.5.4 *If G is a group of order p^3q, then $\Gamma(G)$ is perfect.*

Proof : We first consider the case, when $p > q$. In this case, G has a unique normal Sylow p subgroup of order p^3. We first show that $\Gamma(G)$ has no induced odd cycle of length ≥ 5. Suppose, $\Gamma(G)$ has an induced odd cycle $C : H_1 \sim H_2 \sim \cdots H_t$ with $t \geq 5$.

Claim 1: C has no subgroup of order p^3.

Proof of Claim: Let $|H_1| = p^3$. Now, as H_1 is a maximal subgroup which is normal in G and $H_3, H_4 \not\sim H_1$, we must have $H_3, H_4 \subseteq H_1$. This contradicts that $H_3 \sim H_4$.

From Claim 1, it follows that C has no subgroup of order q, as any subgroup of order q can only be adjacent to a subgroup of order p^3.

Claim 2: C has no subgroup of order p.

Proof of Claim: Let $|H_1| = p$. Then $|H_2| = |H_t| = p^2q$ and $|H_2 \cap H_t| = p^2$. Note that H_2, H_t being groups of order p^2q has unique subgroup K of order p^2. As $|H_2 \cap H_t| = p^2$, K is contained in both H_2 and H_t. Again, $|H_3| \in \{p, pq, p^2, p^2q\}$. We show that none of them can happen.

If $|H_3| = p^2$ and as $H_3 \not\sim H_t$, then $|H_3H_t| = p^2q \cdot p^2/p^2 = p^2q$, i.e., $|H_3 \cap H_t| = p^2$, i.e., $H_3 \subseteq K \subseteq H_2$. This contradicts that $H_3 \sim H_2$. Thus $|H_3| \neq p^2$. Similarly, if $|H_3| = p^2q$, we get $|H_3 \cap H_t| = p^2$, i.e, $H_3 \cap H_t = K = H_3 \cap H_2$. Thus $|H_2H_3| = p^4q^2/p^2 = p^2q^2 < p^3q$. This contradicts that $H_3 \sim H_2$. Thus $|H_3| \neq p^2q$. If $|H_3| = p$ and as $H_3 \not\sim H_t$, we have $H_3 \subseteq H_t$, i.e., $H_3 \subseteq H_2$, contradicting $H_3 \sim H_2$. If $|H_3| = pq$, we have $p \mid |H_3 \cap H_t|$, i.e., $p \mid |H_3 \cap H_2|$. Thus $|H_2H_3| = p^2q^2$ or $p^2q < p^3q$, contradicting $H_3 \sim H_2$. Hence Claim 2 holds.

Claim 3: C has no subgroup of order p^2q.

Proof of Claim: Let $|H_1| = p^2q$. Then $|H_3|, |H_4| \in \{pq, p^2, p^2q\}$. As in Claim 3, we can show that none of them can occur.

Using Claims $1-3$, we can conclude that all the vertices in C are subgroups of order pq or p^2. Moreover, it is evident that subgroups order pq or p^2 must alternate in C. However, as C is an odd cycle, this leads to a contradiction.

Now we consider the case when $p < q$. In this case, any subgroup of order p^2q is both maximal and normal in G. We show that $\Gamma(G)$ has no induced odd cycle of length ≥ 5. Suppose, $\Gamma(G)$ has an induced odd cycle $C : H_1 \sim H_2 \sim \cdots H_t$ with $t \geq 5$.

Claim 4: C has no subgroup of order p^2q.

Proof of Claim: Let $|H_1| = p^2q$, i.e, H_1 is a maximal normal subgroup of G. As H_3, H_4 are not adjacent to H_1, we have $H_3, H_4 \subsetneq H_1$, contradicting that $H_3 \sim H_4$.

From Claim 4, it follows that C has no subgroup of order p, as subgroups of order p can be adjacent only to subgroups of order p^2q.

Claim 5: C has no subgroup of order q.

Proof of Claim: Let $|H_1| = q$. Then $|H_2| = |H_t| = p^3$. As $H_2 \sim H_3$, we must have $|H_3| = q$ or pq. In any case, we get $H_3 \sim H_t$, a contradiction.

If C has a subgroup H_1 or order p^3, then q does not divide $|H_3|, |H_4|$, i.e, $|H_3|, |H_4|$ is p, p^2 or p^3. This contradicts that $H_3 \sim H_4$. Thus C has no subgroup of order p^3.

From above arguments it follows that all the vertices in C are subgroups of order pq or p^2. Moreover, it is evident that subgroups order pq or p^2 must alternate in C. However, as C is an odd cycle, this leads to a contradiction.

Thus for all primes p, q and for any group G of order pq, $\Gamma(G)$ has no induced odd cycle of length ≥ 5. Similarly, it can be shown that $\Gamma^c(G)$ has no induced odd cycle of length ≥ 5. Thus $\Gamma(G)$ is perfect. $\square$

Corollary 2.5.2 *If G is a group of order p, p^2, p^3, pq, pqr or p^2q, then $\Gamma(G)$ is perfect.*

Proof : We consider the case when $|G| = p^2q$. Let $H = G \times \mathbb{Z}_p$. Then H is a group of order p^3q and $G \cong G \times \{0\}$ has a normal complement $\{e\} \times \mathbb{Z}_p \cong \mathbb{Z}_p$ in H. Thus by Lemma 2.5.1 and Theorem 2.5.4, it follows that $\Gamma(G)$ is perfect. Similar proofs follow for other orders. $\square$

Remark 2.5.3 *There exist groups of order $p^4, p^3q^2, p^2q^2r, p^3qr$ for which the comaximal subgroup graph is not perfect. For example, for groups $G = \mathbb{Z}_3 \times S_3 \times S_3$ of order $108 = 2^2 \cdot 3^3$, $G = \mathbb{Z}_3 \times \mathbb{Z}_6 \times D_5$ of order $180 = 2^2 \cdot 3^2 \cdot 5$, $G = \mathbb{Z}_{15} \times D_4$ of order $120 = 2^3 \cdot 3 \cdot 5$, $\Gamma(G)$ is not perfect. In fact, for some particular orders there exists exactly one group upto isomorphism for which the comaximal subgroup graph is not perfect. One such is mentioned in the concluding section.*

2.6 Classification upto Isomorphism

It is observed that non-isomorphic groups may give rise to isomorphic comaximal subgroup graphs. As an example, $\Gamma(\mathbb{Z}_6) \cong \Gamma(\mathbb{Z}_{15}) \cong K_2$. So, it is natural to ask: which groups yield unique comaximal subgroup graphs or under what condition, $\Gamma(G_1) \cong \Gamma(G_2)$ implies $G_1 \cong G_2$ In this section, we show that the comaximal subgroup graph of quaternion group (Q_8) is unique and the comaximal subgroup graph of the alternating group (A_4) is unique upto a class of groups. This leads to a partial answer of our question. We start with some results which will be used later in the section.

Proposition 2.6.1 *If G is a finite group with exactly five subgroups, then $G \cong \mathbb{Z}_{p^4}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Lemma 2.6.2 *Let G be a finite group and H, K be two subgroups of G such that $G = HK$ and $|H \cap K| = 1$. If K is a cyclic normal subgroup of G, then all subgroups of K are also normal in G.*

Proof : Let $K = \langle a \rangle$ be normal in G. Let $K_1 = \langle b \rangle$ be a subgroup of K with $b = a^t$. Note that any element of G can be written uniquely written as ha^i where i is a positive integer and $h \in H$. Now,

$$(ha^i)b^j(ha^i)^{-1} = h(a^i{\cdot}a^{tj}{\cdot}a^{-i})h^{-1} = (ha^jh^{-1})^t = (a^l)^t = (b)^l \in K_1, \text{ i.e., } K_1 \trianglelefteq G.$$

□

As $G = HK$ implies $G = KH$, the role of H and K are interchangeable in the above lemma.

Theorem 2.6.1 *Let G be a finite group such that $\Gamma(G) \cong \Gamma(A_4)$, then $G \cong A_4$ or $\mathbb{Z}_{p^4q}$, where p and q are distinct primes.*

Proof : We start by noting that $\Gamma(A_4)$ is isomorphic to $K_{1,4} \cup 3K_1$. Let G be a finite group such that $\Gamma(G) \cong \Gamma(A_4)$. As independence number of $\Gamma(A_4) = 7 \leq 8$, by Theorem 2.4.1, G must be solvable.

If G is nilpotent, as $\Gamma^*(G)$ is a star and $\Gamma(G)$ has isolated vertices, by Theorem 3.4 [23], $G \cong \mathbb{Z}_{p^rq}$ where $r \geq 2$. Note that in this case, $\Gamma(G)$ has $\langle pq\rangle, \langle p^2q\rangle, \ldots, \langle p^{r-1}q\rangle$ as isolated vertices, i.e., $r-1$ isolated vertices. So $r-1=3$ and hence $G \cong \mathbb{Z}_{p^4q}$.

Now, we assume that G is not nilpotent. As G is solvable, $G = HK$ and $H \cap K = \{e\}$, where H is a Sylow-p-subgroup of G and K is a Hall p'-subgroup of G. Note that K is non-trivial as otherwise G is a p-group, i.e., nilpotent, a contradiction.

Case 1: H is not a unique Sylow p-subgroup of G: Then the number of Sylow p-subgroup of G, $n_p = 1 + pl \geq p + 1$. Note that all the Sylow p-subgroups are adjacent to K in $\Gamma(G)$. Thus K is the universal vertex of $\Gamma^*(G)$ and all the Sylow p-subgroups are pendant vertices, i.e., $p+1 \leq 4$ which implies $p = 2$ or 3. Also note that $n_p \neq 1 + pl$ with $l \geq 2$, as otherwise we get $1 + 2p \leq 4$, a contradiction. Thus $n_p = 1 + p$. Let $H = H_1, H_2, \ldots, H_{p+1}$ be the Sylow p-subgroups of G.

Also note that K is the unique subgroup of order $|K|$ in G, because if K' is another subgroup of order $|K|$ in G, then K' is also adjacent to all the Sylow p-subgroups of G, a contradiction. Thus $K \trianglelefteq G$.

Further note that K is a maximal subgroup of G, because if K' is a proper subgroup of G containing K, then K' is adjacent to all of $H = H_1, H_2, \ldots, H_{p+1}$, a contradiction.

If any of the Sylow p-subgroup H_i has a proper non-trivial subgroup H', then KH' is a proper subgroup of G which properly contains K. Thus H_i's has no proper subgroups, i.e., $|H_i| = p$.

Since K is a maximal subgroup of G which is normal in G, the three isolated vertex in $\Gamma(G)$ corresponds to three non-trivial proper subgroups of K. Thus K has exactly five subgroups and hence, by Proposition 2.6.1, $K \cong \mathbb{Z}_{q^4}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$, where q is a prime. So $|G| = |H||K| = pq^4$ or $4p$.

Case 1a: $|G| = pq^4$: As K is a cyclic normal subgroup of G, by Lemma 2.6.2, all of its proper subgroups K_1, K_2, K_3 of order q, q^2, q^3 respectively are normal in G. Thus $K, K_1, K_2, K_3, H, HK_1, HK_2, HK_3, H_2, \ldots, H_{p+1}$ are distinct proper subgroups of G and this exceeds the order of the graph.

Case 1b: $|G| = 4p$: We already have $p = 2$ or 3. However, as $|K| = 4$ and $gcd(|H|, |K|) = 1$, we must have $p = 3$, i.e., $|G| = 12$. Now there are five groups upto isomorphism of order 12, namely $\mathbb{Z}_{12}, \mathbb{Z}_6 \times \mathbb{Z}_2, A_4, D_6$ and Dicyclic group of order 12. The first two are nilpotent and hence omitted. The last two have 3 subgroups each of order 4, contradicting the fact that K is the unique subgroup of order $|K|$ in G. Thus $G \cong A_4$.

Case 2: H is the unique Sylow p-subgroup of G and hence $H \trianglelefteq G$: As G is solvable, there exists atleast one maximal subgroup of G which is normal in G.

Case 2a: Let there exists a maximal subgroup L other than H, K such that $L \trianglelefteq G$: Clearly both H and K are not contained in L, as $HK = G$. Without loss of generality, let $K \nsubseteq L$. Then $L \sim K \sim H$ and K is the universal vertex in $\Gamma^*(G)$. Also note that $H \subsetneq L$, as otherwise we get a cycle in $\Gamma(G)$. Let $a \in G \setminus (L \cup K)$. Thus $\langle a \rangle \nsubseteq L$, $\langle a \rangle \neq K$ and $\langle a \rangle \sim L$. Thus L also has degree greater than or equal to 2, a contradiction.

Case 2b: H is a maximal subgroup of G and $H \trianglelefteq G$: As $G \neq H \cup K$, choose $a \in G \setminus (H \cup K)$. Thus $\langle a \rangle \sim H$ and we have $H \sim K$, i.e., degree of H is greater than or equal to 2. Hence H is the universal vertex in $\Gamma^*(G)$. As any subgroup not contained in H is adjacent to H and $\Gamma(G)$ has three isolated vertices, we conclude that H has exactly five subgroups and hence, by Proposition 2.6.1, $H \cong \mathbb{Z}_{p^4}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Again as $\Gamma(G)$ has four pendant vertices or leaves, there are exactly four subgroups which are not contained in H, out of which one is K. If K is not cyclic, then K has atleast three proper subgroups K_1, K_2, K_3 respectively. Thus K, K_1, K_2, K_3 and $\langle a \rangle$ are five subgroups not contained in H, a contradiction. Thus K is cyclic.

Let $|K| = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$. Then K has $(\alpha_1 + 1)(\alpha_1 + 1) \cdots (\alpha_1 + 1) - 1$ non-trivial subgroups which are adjacent to H and $\langle a \rangle$ is another subgroup adjacent to H. Thus $(\alpha_1 + 1)(\alpha_1 + 1) \cdots (\alpha_1 + 1) \leq 4$, i.e., $|K| = p_1 p_2, {p_1}^2$ or ${p_1}^3$. In all cases, K has a subgroup of order p_1, say K_1.

If $H \cong \mathbb{Z}_{p^4}$, i.e., if H is cyclic, by Lemma 2.6.2, all subgroups of H are normal in G. Let H_1, H_2, H_3 be subgroups of H of order p, p^2, p^3 respectively. Thus $H, K, H_1, H_2, H_3, H_1K,\ H_2K, H_3K, H_1K_1, H_2K_1, H_3K_1$ are distinct proper non-trivial subgroups of G. But this exceeds the order of $\Gamma(G)$, a contradiction.

Let us consider the case when $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then H has three non-trivial proper subgroups H_1, H_2 and H_3. If $|K| = {p_1}^2$ or ${p_1}^3$, then $|G| = 4p_1^2$ or $4p_1^3$. In any case, the number of Sylow p_1 subgroup, $n_{p_1} = 1 + lp_1 | 4$. This implies that either the Sylow p_1-subgroup is unique, i.e., $K \trianglelefteq G$ or $p_1 = 3$ and there exists 4 Sylow p_1-subgroups of G. If $K \trianglelefteq G$, then $G \cong H \times K$, a direct product of two p-groups. Thus G is nilpotent, a contradiction. Hence $p_1 = 3$ and there exists 4 Sylow p_1-subgroups of G, namely K, K_2, K_3, K_4. Also, as the Sylow p_1-subgroups of G are of order p_1^2 or p_1^3, there exists atleast one subgroup A of K of order p_1. Thus, we get at least nine vertices, namely $H, H_1, H_2, H_3, K, K_2, K_3, K_4, A$, in $\Gamma(G)$, a contradiction.

Thus the only case left is $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $K \cong \mathbb{Z}_{p_1p_2}$. If there exists another subgroup K' of G such that $|K| = |K'|$, then there are atleast 5 subgroups contained in K or K', including K or K'. Also, there are 4 subgroups contained in H, including H. Thus the total count of vertices become at least 9, a contradiction. Thus K is the unique subgroup of order p_1p_2 in G and hence $K \trianglelefteq G$. This implies $G \cong H \times K$ and G is nilpotent, a contradiction.

Case 2c: K is a maximal subgroup of G and $K \trianglelefteq G$: Also as $H \trianglelefteq G$, we have $G \cong H \times K$. As $G \neq H \cup K$, choose $a \in G \setminus (H \cup K)$. Thus $\langle a \rangle \sim K$ and we have $H \sim K$, i.e., degree of K is greater than or equal to

2. Hence K is the universal vertex in $\Gamma^*(G)$. As any non-trivial subgroup not contained in K is adjacent to K, therefore K has exactly 3 non-trvial proper subgroups corresponding to the three isolated points in $\Gamma(G)$. Thus, by Proposition 2.6.1, $K \cong \mathbb{Z}_{p^4}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. In any case, K is a p-group and hence nilpotent. Also H being a Sylow subgroup is also nilpotent, and G being the direct product of H and K is also nilpotent, a contradiction.

Combining all the cases, we get $G \cong A_4$ or $\mathbb{Z}_{p^4q}$, where p and q are distinct primes. □

Theorem 2.6.2 *Let G be a finite group such that $\Gamma(G) \cong \Gamma(Q_8)$, then $G \cong Q_8$.*

Proof : We start by noting that $\Gamma(Q_8) \cong K_3 \cup K_1$. As the independence number of $\Gamma(G)$ is $2 \leq 8$, by Theorem 2.4.1, G is solvable.

As G is solvable, $G = HK$ and $H \cap K = \{e\}$, where H is a Sylow-p-subgroup of G and K is a Hall p'-subgroup of G. If K is the trivial subgroup, then $G = H$ is a p-group with 4 non-trivial proper subgroups and hence nilpotent. Now as $\Gamma^*(G)$ is complete, by Theorem 3.5 of [23], $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or Q_8. As $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is a complete graph on 4 vertices, we have $G \cong Q_8$.

If K is a non-trivial subgroup of G, then $H \sim K$ forms an edge of the triangle in $\Gamma(G)$. If H has two or more proper non-trivial subgroups, we get two or more isolated vertices in $\Gamma(G)$, a contradiction. Thus H has at most one proper non-trivial subgroup, i.e., $H \cong \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$.

If $H \cong \mathbb{Z}_{p^2}$, then H has a unique subgroup H_1 of order p which corresponds to the only isolated vertex in $\Gamma(G)$. Thus K can not have any non-trivial proper subgroup, as that would add another isolated vertex in $\Gamma(G)$. Thus $K \cong \mathbb{Z}_q$ and $|G| = |H|K| = p^2q$. If H is not the unique Sylow p-subgroup of G, then there exists another subgroup L of order p^2 and $K \sim L$. This implies $H \sim L$ and hence $p^2q = |G| = |HL|$. But as H and L are groups of order p^2, $|HL|$ is not divisible by q, a conradiction. Therefore $H \trianglelefteq G$. Thus, by Lemma 2.6.2, $H_1 \trianglelefteq G$. Then H_1K is a non-trivial proper subgroup of G of order pq and $H_1K \sim H$. Therefore from the graph $\Gamma(G)$, we get that $H_1K \sim K$, a contradiction.

If $H \cong \mathbb{Z}_p$, then K can have at most one non-trivial proper subgroup, as each such subgroup corresponds to an isolated vertex in $\Gamma(G)$. Thus $K \cong \mathbb{Z}_q$ or $\mathbb{Z}_{q^2}$. If $K \cong \mathbb{Z}_q$, then $|G| = |H||K| = pq$. By Theorem 3.2 of [23], this implies that $\Gamma(G)$ is a star, a contradiction. If $K \cong \mathbb{Z}_{q^2}$, then $|G| = pq^2$ and K has a unique subgroup K_1 of order q, which corresponds to the isolated vertex in $\Gamma(G)$. If H is not the unique Sylow p-subgroup of G, then there exists another Sylow p-subgroup L of G of order p. Also $KL = G$, i.e., $K \sim L$. Now from the graph $\Gamma(G)$, we have $H \sim L$, i.e., $pq^2 = |G| = |HL|$. But, arguing similarly as above, we get that q does not divide $|HL|$, a contradiction. Thus H is the unique Sylow p-subgroup of G and $H \trianglelefteq G$. As a result, HK_1 is a proper subgroup of G of order pq and $HK_1 \sim K$. Thus, from the graph $\Gamma(G)$, we have $H \sim HK_1$, a contradiction.

Combining the above cases, we see that $G \cong Q_8$. □

Remark 2.6.3 *In a similar way, it has been shown that if for a finite solvable group G, $\Gamma(G) \cong \Gamma(D_{2^k})$ holds, then $G \cong D_{2^k}$, where D_{2^k} is the dihedral* *group of order 2^{k+1}. The proof is long and will appear in the next chapter.*

2.7 Conclusion and Open Issues

In this chapter, we continued the study of co-maximal subgroup graph $\Gamma(G)$ and introduced the deleted co-maximal subgroup graph $\Gamma^*(G)$ of a group G. We discuss its various properties like connectdeness, girth and bipartiteness, independence number, chromatic number, perfectness, isomorphism problems on $\Gamma(G)$. However, there are natural questions which are yet to be resolved.

The first question arises from Remark 2.2.6.

Question 1: Does there exist a finite group G such that $diam(\Gamma^*(G)) = 4$?

On the light of Corollary 2.2.4 and Theorem 2.2 of [6], we can say that if such a group G exists, then it must be non-nilpotent and $\Gamma(G)$ must have isolated vertices.

The next question arises from Remark 2.2.8.

Question 2: Does there exist a finite non-solvable group such that $\Gamma^*(G)$ is disconnected?

Question 3: In Theorem 2.2.5, it was proved that $\Gamma_{max}(G)$ is connected if and only if $\Gamma^*(G)$ is connected. However we need to characterize G for which $\Gamma_{max}(G)$ is connected.

Question 4: In Theorem 2.4.1, it was proved that if independence number of $\Gamma(G)$ is less than or equal to 8, then G is solvable. But, we feel that a stronger bound is true. As A_5 is the smallest non-solvable group and $\alpha(\Gamma(A_5)) = 52$, we strongly believe that: $\alpha(\Gamma(G)) \leq 51 \Rightarrow G$ is solvable?

Question 5: In view of Remark 2.5.3, we conjecture that: If G is a group of order p^4, $\Gamma(G)$ is perfect if and only if $G \ncong \mathbb{Z}_p^4$.

In the previous section, we have shown examples of groups which can be uniquely recovered from their comaximal subgroup graphs, e.g., quaternion group, dihedral groups. It remains open to classify all finite groups G which can be uniquely recovered from their comaximal subgroup graphs.