Chapter 3

Co-Maximal Subgroup Graph of $\mathbb{Z}_n$ and D_n

In this chapter, we study various properties of co-maximal subgroup graph of $\mathbb{Z}_n$ and D_n.

3.1 Co-Maximal Subgroup Graph of $\mathbb{Z}_n$

We start with some basic properties of $\Gamma(\mathbb{Z}_n)$ and $\Gamma^*(\mathbb{Z}_n)$. As for any cyclic p-group G, $\Gamma(\mathbb{Z}_n)$ is empty, throughout the paper, we consider $\Gamma(\mathbb{Z}_n)$ where n is not a prime power.

3.1.1 Basic Properties of $\Gamma(\mathbb{Z}_n)$

In this section, we study some basic properties of $\Gamma(\mathbb{Z}_n)$ and $\Gamma^*(\mathbb{Z}_n)$ like connectedness, degree, diameter etc.

Lemma 3.1.1 *Let $H = \langle x \rangle$ and $K = \langle y \rangle$ be two subgroups of $\mathbb{Z}_n$ where x, y divide n. Then $H \sim K$ in $\Gamma(\mathbb{Z}_n)$ if and only if $gcd(x, y) = 1$.*

Proof : It follows from Bezout's theorem and the observation that $HK = \{sx + ty : s, t \in \mathbb{Z}\}$. $\square$

Theorem 3.1.1 *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$. Let $H = \langle {p_1}^{\beta_1}{p_2}^{\beta_2} \cdots {p_k}^{\beta_k} \rangle$ be a subgroup of $\mathbb{Z}_n$, where $\beta_i \leq \alpha_i$. Then degree of H in $\Gamma(\mathbb{Z}_n)$ is*

$$deg(H) = \begin{cases} 0, & \text{if } \beta_i \neq 0, \forall i, \\ \\ \displaystyle\prod_{j:\beta_j = 0} (\alpha_j + 1) - 1, & \text{otherwise.} \end{cases}$$

Proof : Follows from Lemma 3.1.1. $\square$

Corollary 3.1.2 *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$. Then $\Gamma^*(\mathbb{Z}_n)$ is Eulerian if and only if n is a perfect square.*

Proof : If n is a perfect square, then each α_i is even and by Theorem 3.1.1, degree of every vertex of $\Gamma^*(\mathbb{Z}_n)$ is even and hence $\Gamma^*(\mathbb{Z}_n)$ is Eulerian. If n is not a perfect square, then there exists i such that α_i is odd. Let $H = \langle {p_1}^{\alpha_1} \cdots {p_{i-1}}^{\alpha_{i-1}}{p_{i+1}}^{\alpha_{i+1}} \cdots {p_k}^{\alpha_k} \rangle$. Then by Theorem 3.1.1, $deg(H) = \alpha_i$, which is odd. Thus $\Gamma^*(\mathbb{Z}_n)$ is not Eulerian. $\square$

Theorem 3.1.2 *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$. Then $\Gamma(\mathbb{Z}_n)$ has exactly $\alpha_1\alpha_2 \cdots \alpha_k - 1$ isolated vertices.*

Proof : Since G is a cyclic non p-group of order $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$. Then, $G \cong \mathbb{Z}_n$. By Lemma 3.1.1, $H = \langle p_1p_2\cdots p_k\rangle$ is an isolated vertex in $\Gamma(G)$. Similarly, if x is a multiple of $p_1p_2\cdots p_k$ which divides n, then $\langle x\rangle$ is an isolated vertex in $\Gamma(G)$.

Let $A = \langle a\rangle$ with $a|n$ be a subgroup of G such that A is an isolated vertex in $\Gamma(G)$. As G has a unique subgroup of order corresponding to each factor of n and for any non-trivial proper subgroup H of G, we have $A \nsim H$ in $\Gamma(G)$, we have $gcd(a, m) \neq 1$ for any factor m of $|G| = n$. Thus $p_i|a$ for all i, i.e., a is a multiple of $p_1p_2\cdots p_k$ which divides n.

Hence the number of isolated vertices in $\Gamma(G)$ is $\alpha_1\alpha_2\cdots\alpha_k - 1$. □

Corollary 3.1.3 $\Gamma(\mathbb{Z}_n)$ *is connected if and only if* n *is square-free.*

Proof : Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$. The corollary follows from the fact that $\alpha_1\alpha_2\cdots\alpha_k - 1 = 0$ if and only if n is square-free. □

Theorem 3.1.3 *Let* G *be a cyclic non p-group of order* $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, *where* p_i*'s are distinct primes and* $\alpha_i \geq 1$. *Then* $diam(\Gamma^*(G)) = \begin{cases} 2, & \text{if } k = 2 \\ 3, & \text{if } k \geq 3 \end{cases}$

Proof : It is clear that the number of maximal subgroups of G is k. If $k = 2$, then the vertices of $\Gamma^*(G)$ are $\langle p_1\rangle, \langle {p_1}^2\rangle, \ldots, \langle {p_1}^{\alpha_1}\rangle, \langle p_2\rangle, \langle {p_2}^2\rangle, \ldots, \langle {p_2}^{\alpha_2}\rangle$ and any two non-adjacent vertices always have a common neighbour either $\langle p_1\rangle$ or $\langle p_2\rangle$. Hence its diameter is 2.

If $k \geq 3$, then $\langle p_1p_2\cdots p_{k-1}\rangle$ and $\langle p_2p_3\cdots p_k\rangle$ are non-adjacent vertices in $\Gamma^*(G)$ and they do not have any common neighbour. Thus their distance is greater than 2. Now, as $\mathbb{Z}_n$ is nilpotent, we have $diam(\Gamma^*(G)) = 3$. □

Theorem 3.1.4 *Let G be a cyclic non p-group of order $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$. Then $\Gamma(G)$ has pendant vertices if and only if $\alpha_i = 1$ for some i.*

Proof : Let G be a cyclic group of order $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, where at least one $\alpha_i = 1$, say $\alpha_1 = 1$, i.e., $n = p_1{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$. Then $\langle p_2p_3\cdots p_k\rangle$ is a pendant vertex in $\Gamma(G)$, which is adjacent to $\langle p_1\rangle$.

Conversely, let G be a cyclic group of order $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$ such that $\Gamma(G)$ has at least one pendant vertex. If possible, let $\alpha_i \geq 2$ for all i. Let $H = \langle m\rangle$ be a pendant vertex in $\Gamma(G)$ where $m|n$. If $p_i|m$ for all i, then H is an isolated vertex, a contradiction. Thus, m misses at least one prime factor. Let $m = {p_2}^{\beta_2}\cdots{p_k}^{\beta_k}$ where $0 \leq \beta_i \leq \alpha_i$. But this implies that H is adjacent to the vertices $\langle p_1\rangle, \langle {p_1}^2\rangle, \ldots, \langle {p_1}^{\alpha_1}\rangle$. As $\alpha_1 \geq 2$, H can not be a pendant vertex. Thus, at least some α_i must be 1. □

3.1.2 Hamiltonicity, Perfectness and Dominating Sets of $\Gamma(\mathbb{Z}_n)$

In this section, we characterize the values of n for which $\Gamma^*(\mathbb{Z}_n)$ is perfect and hamiltonian. We also find the domination number of $\Gamma^*(\mathbb{Z}_n)$.

Theorem 3.1.5 *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$. Then $\Gamma^*(\mathbb{Z}_n)$ is Hamiltonian if and only if $k = 2$ and $\alpha_1 = \alpha_2$.*

Proof : If $k = 2$ and $\alpha_1 = \alpha_2$, then $n = {p_1}^{\alpha_1}{p_2}^{\alpha_1}$. We now explicitly construct the hamiltonian circuit in $\Gamma^*(\mathbb{Z}_n)$:

$$\langle p_1\rangle \sim \langle p_2\rangle \sim \langle p_1^2\rangle \sim \langle p_2^2\rangle \sim \langle p_1^3\rangle \sim \langle p_2^3\rangle \sim \cdots \sim \langle p_1^{\alpha_1}\rangle \sim \langle p_2^{\alpha_1}\rangle \sim \langle p_1\rangle.$$

Conversely, let $\Gamma^*(\mathbb{Z}_n)$ be Hamiltonian. If possible, let $k \geq 3$. If $\alpha_i = 1$ for some i, then the graph has a vertex of degree 1 and hence it is not hamiltonian. Thus, we assume that $\alpha_i \geq 2$ for all i. Without loss of generality, let $\alpha_1 = min\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. Now, the vertices of the form $\langle {p_2}^{\alpha_2'} {p_3}^{\alpha_3'} \cdots {p_k}^{\alpha_k'}\rangle$ are adjacent only to the vertices of the form $\langle {p_1}^{\alpha_1'}\rangle$, where $1 \leq \alpha_i' \leq \alpha_i$, i.e., we have $\alpha_2\alpha_3\cdots\alpha_k$ vertices of degree α_1. As two vertices of the form $\langle {p_2}^{\alpha_2'} {p_3}^{\alpha_3'} \cdots {p_k}^{\alpha_k'}\rangle$ are not adjacent, to complete a hamiltonian cycle, we need at least $\alpha_2\alpha_3\cdots\alpha_k$ different vertices between the vertices of the form $\langle {p_2}^{\alpha_2'} {p_3}^{\alpha_3'} \cdots {p_k}^{\alpha_k'}\rangle$. But, as $k \geq 3$, we have $\alpha_2\alpha_3\cdots\alpha_k > \alpha_1$. This leads to a contradiction. Thus $k = 2$ and $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}$.

As earlier, we can assume that $\alpha_1, \alpha_2 \geq 2$. Let, if possible, $\alpha_1 \neq \alpha_2$. Without loss of generality, let $2 \leq \alpha_1 < \alpha_2$. Now, on any hamiltonian circuit in $\Gamma^*(\mathbb{Z}_n)$, between any two vertices of the form $\langle {p_1}^i\rangle$ and $\langle {p_1}^j\rangle$ we have a vertex of the form $\langle {p_2}^t\rangle$ and between any two vertices of the form $\langle {p_2}^i\rangle$ and $\langle {p_2}^j\rangle$ we have a vertex of the form $\langle {p_1}^t\rangle$. Thus any Hamiltonian circuit should consist of an alternating run of vertices of the form $\langle {p_1}^i\rangle$ and $\langle {p_2}^j\rangle$. However, as $\alpha_1 < \alpha_2$, we have more vertices of the form $\langle {p_2}^j\rangle$ than that of the form $\langle {p_1}^i\rangle$, a contradiction. Thus $\alpha_1 = \alpha_2$. □

Theorem 3.1.6 *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \geq 1$. Then $\Gamma^*(\mathbb{Z}_n)$ is perfect if and only if $k \leq 4$.*

Proof : If $k \geq 5$, then there exists an induced 5-cycle in $\Gamma^*(\mathbb{Z}_n)$ as shown in Figure 3.1.2. Thus, in this case, $\Gamma^*(\mathbb{Z}_n)$ is not perfect. Let $k \leq 4$, i.e., n has at most 4 distinct prime factors p_1, p_2, p_3, p_4. Let, if possible, $\Gamma^*(\mathbb{Z}_n)$ admits an induced odd cycle of length $t \geq 5$, say $\langle h_1 \rangle \sim \langle h_2 \rangle \sim \cdots \sim \langle h_t \rangle \sim \langle h_1 \rangle$. From the non-adjacency relations, we get $gcd(h_1, h_3), gcd(h_1, h_4), gcd(h_2, h_4), gcd(h_2, h_5), gcd(h_3, h_t) \neq 1$.

Let $p_1 \mid gcd(h_1, h_3)$. Then $p_1 \mid h_1$ and $p_1 \mid h_3$. Again, as $\langle h_t \rangle \sim \langle h_1 \rangle$, we have $gcd(h_1, h_t) = 1$, i.e., $p_1 \nmid h_t$.

Similarly, as $\langle h_3 \rangle \sim \langle h_4 \rangle$, we have $p_1 \nmid h_4$, i.e., $p_1 \nmid gcd(h_1, h_4)$. Let $p_2 \mid gcd(h_1, h_4)$. Then $p_2 \mid h_1$ and $p_2 \mid h_4$. Now as $\langle h_3 \rangle \sim \langle h_4 \rangle$, we have $p_2 \nmid h_3$.

Again, as $p_1, p_2 \mid h_1$ and $\langle h_1 \rangle \sim \langle h_2 \rangle$, we have $p_1, p_2 \nmid h_2$, i.e., $p_1, p_2 \nmid gcd(h_2, h_4)$. Let $p_3 \mid gcd(h_2, h_4)$. Then $p_3 \mid h_2$ and $p_3 \mid h_4$. As $\langle h_2 \rangle \sim \langle h_3 \rangle$, we have $p_3 \nmid h_3$.

Thus $p_1, p_2, p_3 \nmid gcd(h_3, h_t)$. Let $p_4 \mid gcd(h_3, h_t)$. Then $p_4 \mid h_3$ and $p_4 \mid h_t$. As $\langle h_2 \rangle \sim \langle h_3 \rangle$, we have $p_4 \nmid h_2$. Again, as $\langle h_4 \rangle \sim \langle h_5 \rangle$, we have $p_3 \nmid h_5$.

From the above situation, we get $p_1, p_2, p_3, p_4 \nmid gcd(h_2, h_5)$. This is a contradiction, as $gcd(h_2, h_5) \neq 1$ and $k \leq 4$. Thus $\Gamma^*(\mathbb{Z}_n)$ does not admit any induced odd cycle of length $t \geq 5$.

Let, if possible, $\Gamma^*(\mathbb{Z}_n)^c$ admits an induced odd cycle of length $t \geq 5$, say $\langle h_1 \rangle \sim \langle h_2 \rangle \sim \cdots \sim \langle h_t \rangle \sim \langle h_1 \rangle$. Note that in the complement graph, two vertices $\langle h_i \rangle$ and $\langle h_j \rangle$ are non-adjacent/adjacent according as $gcd(h_i, h_j)$ is equal/not equal to 1 respectively.

As $\langle h_1 \rangle \sim \langle h_2 \rangle$, we have $gcd(h_1, h_2) \neq 1$. Let $p_1 \mid gcd(h_1, h_2)$. Then $p_1 \mid h_1$ and $p_1 \mid h_2$. As $gcd(h_1, h_3) = 1$, we have $p_1 \nmid h_3$, i.e., $p_1 \nmid gcd(h_2, h_3)$. Similarly, we can conclude that p_1 does not divide any one of $gcd(h_3, h_4), gcd(h_4, h_5), gcd(h_1, h_t)$.

Let $p_2 \mid gcd(h_2, h_3)$. Then $p_2 \mid h_2$ and $p_2 \mid h_3$. As $gcd(h_2, h_4) = 1$, we have $p_2 \nmid h_4$, i.e., p_2 does not divide $gcd(h_3, h_4)$ and $gcd(h_4, h_5)$. Similarly, as $gcd(h_2, h_t) = 1$, we have $p_2 \nmid h_t$, i.e., $p_2 \nmid gcd(h_1, h_t)$.

As $p_1, p_2 \nmid gcd(h_3, h_4)$, let $p_3 \mid gcd(h_3, h_4)$. Then $p_3 \mid h_3$ and $p_3 \mid h_4$. As $gcd(h_1, h_3) = 1$, we have $p_3 \nmid h_1$, i.e., $p_3 \nmid gcd(h_1, h_t)$. Similarly, as $gcd(h_3, h_5) = 1$, we have $p_3 \nmid h_5$, i.e., $p_3 \nmid gcd(h_4, h_5)$.

As $p_1, p_2, p_3 \nmid gcd(h_4, h_5)$, let $p_4 \mid gcd(h_4, h_5)$. Then $p_4 \mid h_4$ and $p_4 \mid h_5$. As $gcd(h_1, h_4) = 1$, we have $p_4 \nmid h_1$, i.e., $p_4 \nmid gcd(h_1, h_t)$.

Thus $p_1, p_2, p_3, p_4 \nmid gcd(h_1, h_t)$. But this is a contradiction, as $gcd(h_1, h_t) > 1$ and n has at most four distinct prime factors. Thus $\Gamma^*(\mathbb{Z}_n)^c$ does not admit an induced odd cycle of length $t \geq 5$.

Hence, by strong perfect graph theorem, the theorem follows. $\square$

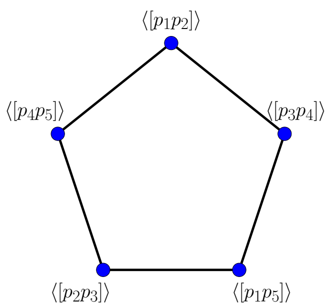

Figure 3.1: Induced 5-cycle in $\Gamma^*(\mathbb{Z}_n)$, for $k \geq 5$

Theorem 3.1.7 *Let* $n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$*, where* p_i*'s are distinct primes,*

$k \geq 2$ *and* $\alpha_i \geq 1$. *Then*

$$\gamma(\Gamma^*(\mathbb{Z}_n)) = \begin{cases} 1, & \text{if } n = {p_1}^{\alpha_1} p_2. \\ k, & \text{otherwise.} \end{cases}$$

Proof : Clearly $\{\langle p_1\rangle, \langle p_2\rangle, \ldots, \langle p_k\rangle\}$ is a dominating set for $\Gamma^*(\mathbb{Z}_n)$ of size k. Thus $\gamma(\Gamma^*(\mathbb{Z}_n)) \leq k$.

Let $S = \{\langle x_1\rangle, \langle x_2\rangle, \ldots, \langle x_{k-1}\rangle\}$ be a dominating set of $\Gamma^*(\mathbb{Z}_n)$ of size $k-1$. Let $m = p_1 p_2 p_3 \cdots p_k$. Out of the k vertices $\langle m/p_1\rangle, \langle m/p_2\rangle, \ldots, \langle m/p_k\rangle$, at least one does not belong to S. Without loss of generality, let $\langle m/p_1\rangle \notin S$ and $\langle m/p_1\rangle \sim \langle x_1\rangle$. Thus, by Lemma 3.1.1, $x_1 = p_1^{\alpha'_1}$, where $1 \leq \alpha'_1 \leq \alpha_1$. Thus $\langle x_1\rangle$ is not adjacent to any of the $k-1$ vertices $\langle m/p_2\rangle, \langle m/p_3\rangle, \ldots, \langle m/p_k\rangle$. Again, by similar argument, not all of these $k-1$ vertices belong to S. Without loss of generality, let $\langle m/p_2\rangle \notin S$ and $\langle m/p_2\rangle \sim \langle x_2\rangle$. Proceeding similarly, we get $x_2 = p_2^{\alpha'_2}$, where $1 \leq \alpha'_2 \leq \alpha_2$. Thus $\langle x_1\rangle$ and $\langle x_2\rangle$ are not adjacent to any of the $k-2$ vertices $\langle m/p_3\rangle, \ldots, \langle m/p_k\rangle$. Continuing in this way, we get $x_i = p_i^{\alpha'_i}$ for $i = 1, 2, \ldots, k-1$. However, in that case, $\langle m/p_k\rangle$ neither belong to S nor adjacent to any element of S, a contradiction. Hence $\gamma(\Gamma^*(\mathbb{Z}_n)) = k$.

Note that the proof does not work if $k = 2$ and exactly one of the two powers is 1. Because in that case, one of $\langle m/p_1\rangle$ and $\langle m/p_2\rangle$ is not a vertex of $\Gamma^*(\mathbb{Z}_n)$, i.e., an isolated vertex of $\Gamma(\mathbb{Z}_n)$. If $k = 2$ and $n = {p_1}^{\alpha_1} p_2$, then $\langle p_2\rangle$ dominates $\Gamma^*(\mathbb{Z}_n)$. $\square$

3.1.3 Isomorphisms

In this section, we discuss the conditions under which co-maximal subgroup graphs defined over different cyclic groups are isomorphic. For that, we start with the following definition.

Definition 3.1.1 *Two positive integers n and m are said to be of same prime-factorization type if $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$ and $m = {q_1}^{\beta_1}{q_2}^{\beta_2}\cdots{q_k}^{\beta_k}$ where p_i, q_i's are primes and there exists $\sigma \in S_k$ such that $\alpha_i = \beta_{\sigma(i)}$ for $i = 1, 2, \ldots, k$.*

Theorem 3.1.8 *Let n and m be two integers. Then $\Gamma(\mathbb{Z}_n) \cong \Gamma(\mathbb{Z}_m)$ if and only if m and n are of same prime-factorization type.*

Proof : If m and n are of same prime-factorization type, then the result is obvious. Let $\Gamma(\mathbb{Z}_n) \cong \Gamma(\mathbb{Z}_m)$, then as their clique numbers are equal, both m and n have same number of distinct prime factors. Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$ and $m = {q_1}^{\beta_1}{q_2}^{\beta_2}\cdots{q_k}^{\beta_k}$. Also as they have same number of isolated vertices, we have $\alpha_1 \cdot \alpha_2 \cdots \alpha_k = \beta_1 \cdot \beta_2 \cdots \beta_k$.

Without loss of generality, let $\alpha_1 = min\{\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_k\}$. If possible, let $\alpha_1 \notin \{\beta_1, \beta_2, \ldots, \beta_k\}$. Now, note that any vertex of the form $\langle {p_2}^{\alpha'_2}{p_3}^{\alpha'_3}\cdots{p_k}^{\alpha'_k}\rangle$ $(1 \leq \alpha'_i \leq \alpha_i)$ in $\Gamma(\mathbb{Z}_n)$ is adjacent to only α_1 vertices, namely $\langle p_1\rangle, \langle p_1^2\rangle, \ldots, \langle p_1^{\alpha_1}\rangle$. Thus $\Gamma(\mathbb{Z}_n)$ has $\alpha_2\alpha_3\cdots\alpha_k$ vertices of degree α_1. As $\alpha_1 \leq min\{\beta_1, \beta_2, \ldots, \beta_k\}$ and $\alpha_1 \notin \{\beta_1, \beta_2, \ldots, \beta_k\}$, from Theorem 3.1.1, it follows that $\Gamma(\mathbb{Z}_m)$ has no vertex of degree α_1, a contradiction. Thus $\alpha_1 = \beta_i$ for some i. By suitable renaming, let $\alpha_1 = \beta_1$.

Again, without loss of generality, let $\alpha_2 = min\{\alpha_2, \ldots, \alpha_k, \beta_2, \ldots, \beta_k\}$. If possible, let $\alpha_2 \notin \{\beta_2, \ldots, \beta_k\}$. If $\alpha_2 \neq \beta_1$, then by similar argument, $\Gamma(\mathbb{Z}_m)$ has no vertex of degree α_2, a contradiction. Thus, we assume that $\alpha_2 = \alpha_1 = \alpha_2$. Then $\Gamma(\mathbb{Z}_n)$ has $\alpha_2\alpha_3 \cdots \alpha_k + \alpha_1\alpha_3 \cdots \alpha_k$ of degree α_1 and $\Gamma(\mathbb{Z}_m)$ has $\beta_2\beta_3 \cdots \beta_k$ of degree α_1. As $\Gamma(\mathbb{Z}_n) \cong \Gamma(\mathbb{Z}_m)$, we have

$$\alpha_2\alpha_3 \cdots \alpha_k + \alpha_1\alpha_3 \cdots \alpha_k = \beta_2\beta_3 \cdots \beta_k,$$

$$\text{i.e., } \alpha_3 \cdots \alpha_k(\alpha_1 + \alpha_2) = \frac{\alpha_1 \cdot \alpha_2 \cdots \alpha_k}{\beta_1} \text{ (as } \alpha_1 \cdot \alpha_2 \cdots \alpha_k = \beta_1 \cdot \beta_2 \cdots \beta_k)$$

$$\text{i.e., } \beta_1(\alpha_1 + \alpha_2) = \alpha_1\alpha_2, \text{i.e., } 2\alpha_1^2 = \alpha_1^2, \text{a contradiction.}$$

Thus $\alpha_2 = \beta_i$ for some $i \in \{2, 3, \ldots, k\}$. By suitable renaming, let $\alpha_2 = \beta_2$.

Proceeding this way, suppose in the $(l-1)$-th step, we get $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, l-1$. Without loss of generality, let $\alpha_l = min\{\alpha_l, \ldots, \alpha_k, \beta_l, \ldots, \beta_k\}$. If possible, let $\alpha_l \notin \{\beta_l, \ldots, \beta_k\}$. If $\alpha_l \notin \{\beta_1, \beta_2, \ldots, \beta_{l-1}\}$, then by similar argument, $\Gamma(\mathbb{Z}_m)$ has no vertex of degree α_l, a contradiction. Thus, we assume that $\alpha_l \in \{\beta_1, \beta_2, \ldots, \beta_{l-1}\}$. Let $\alpha_l = \beta_p = \beta_{p+1} = \cdots = \beta_{l-1} = \alpha_p = \alpha_{p+1} = \cdots = \alpha_{l-1}$ for some $1 \leq p \leq l-1$.

Therefore, $\Gamma(\mathbb{Z}_n)$ has

$$(\alpha_1\alpha_2 \cdots \alpha_{p-1}\alpha_{p+1} \cdots \alpha_k) + (\alpha_1 \cdots \alpha_p\alpha_{p+1} \cdots \alpha_k) + \cdots + (\alpha_1 \cdots \alpha_{l-1}\alpha_{l+1} \cdots \alpha_k)$$

$$= \alpha_1 \cdots \alpha_k \left(\frac{1}{\alpha_p} + \frac{1}{\alpha_{p+1}} + \cdots + \frac{1}{\alpha_l}\right) \text{ vertices of degree } \alpha_l.$$

Similarly, $\Gamma(\mathbb{Z}_m)$ has

$$\beta_1 \cdots \beta_k \left(\frac{1}{\beta_p} + \frac{1}{\beta_{p+1}} + \cdots + \frac{1}{\beta_{l-1}} \right) \text{ vertices of degree } \alpha_l.$$

Now, as $\Gamma(\mathbb{Z}_n) \cong \Gamma(\mathbb{Z}_m)$, we have

$$\alpha_1 \cdots \alpha_k \left(\frac{1}{\alpha_p} + \frac{1}{\alpha_{p+1}} + \cdots + \frac{1}{\alpha_l} \right) = \beta_1 \cdots \beta_k \left(\frac{1}{\beta_p} + \frac{1}{\beta_{p+1}} + \cdots + \frac{1}{\beta_{l-1}} \right)$$

i.e., $\left(\frac{1}{\alpha_p} + \frac{1}{\alpha_{p+1}} + \cdots + \frac{1}{\alpha_l} \right) = \left(\frac{1}{\alpha_p} + \frac{1}{\alpha_{p+1}} + \cdots + \frac{1}{\alpha_{l-1}} \right) \Rightarrow \left(\frac{l-p+1}{\alpha_l} \right) = \left(\frac{l-p}{\alpha_l} \right.$

a contradiction. Thus, by suitable renaming, we get $\alpha_l = \beta_l$, and hence by induction, the theorem follows. □

3.2 Co-Maximal Subgroup Graph of D_n

In this section, we study the comaximal subgroup graph on finite dihedral groups, denoted by $\Gamma(D_n)$.

3.2.1 Structural Properties of $\Gamma(D_n)$ and $\Gamma^*(D_n)$

We characterize various structural properties of $\Gamma(D_n)$ and $\Gamma^*(D_n)$ of like order, maximum and minimum degree, girth, diameter and when they are Eulerian. We start by describing the complete list of subgroups of D_n, which constitute the vertex set of the graph to be studied.

The dihedral group D_n has two generators r and s with orders n and 2 such that $srs^{-1} = r^{-1}$. $D_n = \langle r, s : r^n = s^2 = 1, srs = r^{n-1} \rangle$ consists of $2n$

elements. We recall a result on the complete list of subgroups of D_n. For a proof of this listing, please refer to [22].

Proposition 3.2.1 *Every subgroup of D_n is either cyclic or dihedral. A complete listing of the subgroups is as follows:*

1. *$\langle r^d \rangle$, where $d|n$, with index $2d$,*

2. *$\langle r^d, r^i s \rangle$, where $d|n$ and $0 \leq i \leq d-1$, with index d.*

Moreover, every subgroup of D_n occurs exactly once in this listing.

Proposition 3.2.2 *$\Gamma(D_n)$ has $\sigma(n) + \tau(n) - 2$ vertices.*

Proof : $\Gamma(D_n)$ contains all subgroups of the form $\langle r^d \rangle$, where $d|n$ and $d \neq n$. We call this vertices of Type-I, and so number of Type-I vertices is $\tau(n) - 1$. Similarly, $\Gamma(D_n)$ contains all subgroups of the form $\langle r^d, r^i s \rangle$, where $d|n$ and $0 \leq i \leq d-1$ except $d = 1$. We call this vertices of Type-II, and so number of Type-II vertices is $\sigma(n) - 1$. □

Now, we investigate the adjacency between vertices of $\Gamma(D_n)$. It is clear that no two vertices of Type-I are adjacent. Thus, any edge of $\Gamma(D_n)$ occurs either between two vertices of Type-II or one of Type-I and one of Type-II. The edges in $\Gamma(D_n)$ are completely classified in the next theorem.

Theorem 3.2.1 *The following are the edges of $\Gamma(D_n)$:*

- *A vertex $\langle r^{d_1} \rangle$ of Type-I is adjacent to a vertex $\langle r^{d_2}, r^i s \rangle$ of Type-II if and only if $gcd(d_1, d_2) = 1$.*

- *Two vertices $\langle r^{d_1}, r^i s\rangle$ and $\langle r^{d_2}, r^j s\rangle$ of Type-II are adjacent if and only if one of the two conditions hold:*

  1. *$gcd(d_1, d_2) = 1$.*
  2. *$gcd(d_1, d_2) = 2$ and $i - j$ is odd.*

Proof :

- Let $H = \langle r^{d_1}\rangle$ and $K = \langle r^{d_2}, r^i s\rangle$. We start by noting that $HK = D_n$ if and only if $r \in HK$. If $gcd(d_1, d_2) = 1$, then there exist integers u, v such that $ud_1 + vd_2 = 1$. Thus, $r = (r^{d_1})^u \cdot (r^{d_2})^v \in HK$. Conversely, as $r \notin H, K$, but $r \in HK$, we must get r as product of powers of r^{d_1} and r^{d_2}, i.e., $gcd(d_1, d_2) = 1$.

- Let $H = \langle r^{d_1}, r^i s\rangle$, $K = \langle r^{d_2}, r^j s\rangle$ and $H \sim K$. Then $HK = D_n$. If $d = gcd(d_1, d_2)$, then there exist integers x, y such that $d_1 x + d_2 y = d$, i.e., $r^d = (r^{d_1})^x (r^{d_2})^y \in HK = D_n$. Thus $\langle r^d\rangle \subseteq HK$. Note that r^d is the smallest power of r that can expressed as product of powers of r^{d_1} and r^{d_2}. If $d \geq 3$, then r and r^2 must be expressible as products of powers of $r^{d_1}, r^i s, r^{d_2}$ and $r^j s$, i.e., there exist integers x_1, x_2, y_1, y_2 such that

  $$d_1x_1 + d_2x_2 + (i-j) \equiv 1 \ (mod\ n) \ \text{ and } d_1y_1 + d_2y_2 + (i-j) \equiv 2 \ (mod\ n).$$

  Subtracting, we get $d_1 u + d_2 v \equiv 1 \ (mod\ n)$, i.e., d divides $d_1 u + d_2 v - 1$, i.e., $d|1$, a contradiction. Thus $d = 1$ or 2. If $d = 1$, we are done. Suppose $d = 2$ and

$i-j$ is even. Note that $d=2$ implies n is even. Now, as $r \in HK$, there exist integers x and y such that $d_1x+d_2y+(i-j) \equiv 1 \ (mod\ n)$. But, $d_1x+d_2y+(i-j)$ is even and it can not be congruent to 1 modulo an even number n. Thus $i-j$ must be odd.

Conversely, let one of the conditions hold. If $d=1$, then any integer can be expressed as integer linear combination of d_1 and d_2. Thus for any integer l, we have $r^l, r^l s \in HK$, i.e., $HK = D_n$. If $d=2$ and $i-j$ is odd, then n is even. As $d=2$, r^2 and all even powers of r can expressed as product of powers of r^{d_1} and r^{d_2} and they belong to HK. For odd powers of r to be in HK, we must have integers x, y such that

$$r^{d_1x+d_2y+(i-j)} = r^{2t+1}, \text{ i.e., } d_1x+d_2y+(i-j) \equiv 2t+1 \ (mod\ n)$$

$$2u = 2t+1+j-i \ (mod\ n)$$

Note that as $gcd(d_1,d_2)=d=2$, for any integer u, we can find x and y such that $d_1x+d_2y=2u$. Also, $2t+1+j-i$ is even. Thus, we have

$$u = \frac{2t+1+j-i}{2} \ (mod\ n)$$

Hence for all values of t, u has a solution and all odd powers of r lies in HK, i.e., $\langle r \rangle \subseteq HK$.

Again, note that $r^{d_1x+d_2y+i}s, r^{d_1x+d_2y+j}s \in HK$ for all values of x, y, i.e., $r^{2l+i}s, r^{2l+j}s \in HK$ for all value of l. As $i-j$ is odd, i and j has different parity, and hence by varying l suitably, all the elements of the form $r^k s \in HK$. Thus $HK = D_n$, i.e., $H \sim K$.

□

In the next few theorems, we find the maximum and minimum degree of $\Gamma(D_n)$, and its number of isolated and pendant vertices.

Theorem 3.2.2 *The maximum degree of* $\Gamma(D_n)$ *is* $\sigma(n) - 1$ *and is attained by* $\langle r \rangle$.

Proof : Among Type-I vertices, $\langle r \rangle$ has the maximum degree and its degree is

$$\left(\sum_{d|n, d \neq 1} d \right) - 1 = \sigma(n) - 1.$$

We claim that the degree of any Type-II vertex is less than $\sigma(n) - 1$.

Case 1: (n is odd, say $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are odd primes). Let $H = \langle r^d, r^i s \rangle$ be a Type-II vertex with $d|n, d \neq 1$. Without loss of generality, let p_1 be a prime divisor of d. Set $K = \langle r^d, s \rangle$ and $L = \langle r^{p_1}, s \rangle$. Clearly $K \subseteq L$. As n is odd, d is also odd. Thus we have

set of neighbours of $H =$ set of neighbours of $K \subseteq$ set of neighbours of L.

Thus $deg(H) = deg(K) \leq deg(L)$. Consider the following two set of vertices

$$A = \{\langle r^{d_1}, s \rangle : p_1 | d_1, d_1 | n\} \text{ and } B = \{\langle r^{d_1} \rangle : p_1 \nmid d_1\}.$$

It is easy to check that all vertices in A are non-adjacent with L and B is the exactly the set of vertices of Type-I which are adjacent to L. Note that

$|A| = \alpha_1(\alpha_2+1)\cdots(\alpha_k+1)$ and $|B| = (\alpha_2+1)\cdots(\alpha_k+1)$. As there are total $(\sigma(n)-1)$ many Type-II vertices and $|A| \geq |B|$, we have

$$deg(H) \leq deg(L) \leq (\sigma(n)-2) - |A| + |B| \leq \sigma(n) - 2 < \sigma(n) - 1.$$

Case 2: (n is even, say $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where $p_1 = 2$ and other p_i's are odd primes). Let $H = \langle r^d, r^i s\rangle$ be a Type-II vertex with $d|n, d \neq 1$ and p_j be a prime divisor of n. According as i is even or odd, set $K = \langle r^d, s\rangle$ or $\langle r^d, rs\rangle$ respectively, and $L = \langle r^{p_j}, s\rangle$ or $\langle r^{p_j}, rs\rangle$, respectively. As in Case 1, we have $deg(H) = deg(K) \leq deg(L)$. Again, as in Case 1, construct the sets A and B. The rest follows similarly and $deg(H) < \sigma(n) - 1$. Thus the theorem follows. □

Theorem 3.2.3 *Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$. The number of isolated vertices in $\Gamma(D_n)$ is $\alpha_1\alpha_2\cdots\alpha_k - 1$. Moreover, $\Gamma(D_n)$ is connected if and only if n is square-free.*

Proof : Note that Type-II vertices are never isolated as they are always adjacent to $\langle r\rangle$. A Type-I vertex $\langle r^d\rangle$ is isolated if and only if $p|d$, for all primes $p|n$, i.e., if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, then the number of isolated vertices are $\alpha_1\alpha_2\cdots\alpha_k - 1$.

As D_n is solvable, it is connected if and only if it has no isolated vertex if and only if $\alpha_1\alpha_2\cdots\alpha_k - 1 = 0$ if and only if n is square-free. □

Theorem 3.2.4 *The minimum degree of* $\Gamma^*(D_n)$ *is given by*

$$\delta(\Gamma^*(D_n)) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

Proof : If n is odd, then $\langle s \rangle$ is adjacent only to $\langle r \rangle$, and hence $\delta = 1$. If n is even, then $\langle s \rangle$ is adjacent only to $\langle r \rangle$ and $\langle r^2, rs \rangle$. Thus degree of $\langle s \rangle$ is 2. We need to show that no vertex have degree 1. Note that every Type-II vertex is adjacent to $\langle r \rangle$ and exactly one of $\langle r^2, s \rangle$ and $\langle r^2, rs \rangle$, i.e., degree of a Type-II vertex is ≥ 2. Let $\langle r^d \rangle$ be a non-isolated Type-I vertex. Then d misses atleast one prime factor of n, say p. Then $\langle r^d \rangle$ is adjacent to $\langle r^p, s \rangle$ and $\langle r^p, rs \rangle$, i.e., its degree is ≥ 2. □

Corollary 3.2.3 *Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ *be odd. The number of pendant vertices in* $\Gamma(D_n)$ *is*

$$p_1 p_2 \cdots p_k \prod_{i=1}^{k} \frac{(p_i{}^{\alpha_i} - 1)}{(p_i - 1)}$$

Proof : If n is even, by Theorem 3.2.4, the minimum degree is 2 and hence $\Gamma(D_n)$ has no pendant vertex. So, we assume that n is odd.

We start by observing that Type-I vertices of the form $\langle r^d \rangle$ are never pendant, as if $\langle r^d \rangle \sim \langle r^x, r^i s \rangle$, then $\langle r^d \rangle \sim \langle r^x, r^j s \rangle$ for $j \neq i$. Thus Type-II vertices are the only possible choices for pendant vertices.

Let $\langle r^d, r^i s \rangle$ be a pendant vertex. If $p_i \nmid d$ for some i, then $\langle r^d, r^i s \rangle$ is adjacent to at least two vertices, namely $\langle r \rangle$ and $\langle r^{p_i} \rangle$. Thus $p_i | d$ for all i.

Finally, if $p_i | d$ for all i, then it is easy to observe that $\langle r^d, r^i s \rangle$ is adjacent only to $\langle r \rangle$. Now, the corollary follows by counting the number of such vertices. □

Proposition 3.2.4 *The girth of* $\Gamma(D_n)$ *is* 3 *for* $n \geq 3$ *and* n *is not an odd prime power.*

Proof : If n is even, then $\langle r \rangle, \langle r^2, s \rangle$ and $\langle r^2, rs \rangle$ forms a triangle. If n is odd, but not a prime power, then there exist two distinct prime factors, say p, q of n. Then $\langle r \rangle, \langle r^p, s \rangle$ and $\langle r^q, s \rangle$ forms a triangle. □

Proposition 3.2.5 $\Gamma^*(D_n)$ *is a star if and only if* n *is an odd prime power.*

Proof : Let $n = p^k$ where p is an odd prime. Then all Type-I vertices except $\langle r \rangle$ are isolated in $\Gamma(D_n)$ and $\langle r \rangle$ is an universal vertex in $\Gamma^*(D_n)$. Now, as any Type-II vertex is of the form $\langle r^{p^l}, r^i s \rangle$, no two of them are adjacent and hence $\Gamma^*(D_n)$ is a star.

Conversely, if $\Gamma^*(D_n)$ is a star and n is not an odd prime power, by above Proposition, $\Gamma(D_n)$ has a triangle, a contradiction. □

As D_n is a finite solvable group, by Theorem 2.2.4, $\Gamma^*(D_n)$ is connected and its diameter is less than or equal to 4. In the next theorem, we compute the diameter of $\Gamma^*(D_n)$ and show that it is either 2 or 3.

Theorem 3.2.5

$$Diam(\Gamma^*(D_n)) = \begin{cases} 2, & n = p^k \\ 3, & else \end{cases}$$

Proof : If n is an odd prime power, by Proposition 3.2.5, $\Gamma^*(D_n)$ is a star and hence $Diam$ $(\Gamma^*(D_n)) = 2$. If $n = 2^k$, then by Theorem 3.6 [23], $\Gamma^*(D_n)$ has an universal vertex and hence $Diam(\Gamma^*(D_n)) = 2$.

If n is not a prime power, then n has at least two distinct prime factors. Let $n = p^\alpha q^\beta m$, where m is coprime to p and q. Then consider the vertices $A = \langle r^{p^a} \rangle$ and $B = \langle r^{n/p^a} \rangle$. Clearly they are non-adjacent. As both are Type-I vertices, if they have a common neighbour, it must be a Type-II vertex, say $\langle r^d, r^i s \rangle$. But that means $d|n, d \neq 1$ and d is coprime to both p^a and n/p^a, a contradiction. Thus A and B have no common neighbour, i.e., $d(A, B) > 2$. Consider the path $\langle r^{p^a} \rangle \sim \langle r^q, s \rangle \sim \langle r^p, s \rangle \sim \langle r^{n/p^a} \rangle$ and hence $d(A, B) = 3$.

We claim that any two vertices are atmost at distance 3 from the other. If both the vertices are of Type-II, then they always have a common neighbour $\langle r \rangle$ and hence their distance is atmost 2. If both are of Type-I and are not isolated, say $\langle r^{d_1} \rangle$ and $\langle r^{d_2} \rangle$, then both d_1 and d_2 miss at least one prime factor of n, say p and q. If $p \neq q$, then $\langle r^{d_1} \rangle \sim \langle r^p, s \rangle \sim \langle r^q, s \rangle \sim \langle r^{d_2} \rangle$, i.e., their distance is atmost 3. If $p = q$, then $\langle r^{d_1} \rangle \sim \langle r^p, s \rangle \sim \langle r^{d_2} \rangle$, i.e., their distance is at most 2. Thus we are left with the case where one of the vertex is of Type-I and other is of Type-II, say $\langle r^{d_1} \rangle$ and $\langle r^{d_2}, r^i s \rangle$. As $\langle r^{d_1} \rangle$ is not isolated, d_1 misses at least one prime factor of n, say p. Thus $\langle r^{d_1} \rangle \sim \langle r^p, s \rangle \sim \langle r \rangle \sim \langle r^{d_2}, r^i s \rangle$, i.e., their distance is at most 3. Hence the theorem follows. □

In the next theorem, we check when $\Gamma^*(D_n)$ is Eulerian.

Theorem 3.2.6 $\Gamma^*(D_n)$ *is Eulerian if and only if* n *is even and all odd prime factors of* n *are of even exponent.*

Proof : Let $\Gamma^*(D_n)$ be Eulerian. If n is odd, by Theorem 3.2.4, minimum degree is 1, i.e., odd, a contradiction. So n must be even. Let n has an odd prime factor p of odd exponent α, i.e., $n = p^\alpha m$, where m is even and $p \nmid m$. Consider the vertex $\langle r^m \rangle$. Observe that its only neighbours are of the form $\langle r^{p^*}, r^i s\rangle$. Thus degree of $\langle r^m \rangle$ is $p + p^2 + \cdots + p^\alpha$, i.e., odd, a contradiction. Hence all odd prime factors of n are of even exponent.

Conversely, let n be even and all odd prime factors of n are of even exponent. Let $n = 2^\alpha {p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$, where α_i's are even. We will show that all non-isolated vertices have even degree.

Let us first consider the Type-I vertices of the form $\langle r^d \rangle$. If d is divisible by all the prime factors of n, then $\langle r^d \rangle$ is an isolated vertex. So, we assume that d is not divisible by some prime factors of n. Suppose $p_{i_1}, p_{i_2}, \ldots, p_{i_t}$ are the prime factors of n not dividing d. Then the neighbours of $\langle r^d \rangle$ are of the form $\langle r^{{p_{i_1}}^{\beta_1} {p_{i_2}}^{\beta_2} \cdots {p_{i_t}}^{\beta_t}}, r^j s\rangle$, where not all β_i's are zero simultaneously. Thus degree of $\langle r^d \rangle$ is $\sigma({p_{i_1}}^{\alpha_{i_1}} {p_{i_2}}^{\alpha_{i_2}} \cdots {p_{i_t}}^{\alpha_{i_t}}) - 1$, which is even, as each α_i is even. Thus Type-I vertices are of even degree.

Now, we consider the Type-II vertices of the form $\langle r^d, r^i s\rangle$. If d is divisible by all the prime factors of n, then $\langle r^d, r^i s\rangle$ has precisely two neighbours, $\langle r \rangle$ and exactly one of $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$. So, we assume that d is not divisible by some prime factors of n. Suppose $p_{i_1}, p_{i_2}, \ldots, p_{i_t}$ are the prime factors of n not dividing d.

Case 1: $(2 \nmid d)$ In this case, the neighbours of $\langle r^d, r^i s\rangle$ are of the form $\langle r^{{p_{i_1}}^{\beta_1}{p_{i_2}}^{\beta_2}\cdots{p_{i_t}}^{\beta_t}}\rangle$ and $\langle r^{{p_{i_1}}^{\beta_1}{p_{i_2}}^{\beta_2}\cdots{p_{i_t}}^{\beta_t}}, r^j s\rangle$ where not all β_i's are zero. Thus the degree of $\langle r^d, r^i s\rangle$ is

$$\tau({p_{i_1}}^{\alpha_{i_1}}{p_{i_2}}^{\alpha_{i_2}}\cdots{p_{i_t}}^{\alpha_{i_t}}) + \sigma({p_{i_1}}^{\alpha_{i_1}}{p_{i_2}}^{\alpha_{i_2}}\cdots{p_{i_t}}^{\alpha_{i_t}}) - 2,$$

which is even, as explained earlier.

Case 2: $(2|d)$ In this case, apart from the neighbours mentioned in Case 1, $\langle r^d, r^i s\rangle$ has neighbours of the form $\langle r^{2^{\beta}{p_{i_1}}^{\beta_1}{p_{i_2}}^{\beta_2}\cdots{p_{i_t}}^{\beta_t}}, r^j s\rangle$, where $i - j$ is odd. However, proceeding similarly as above, it can be shown that the number of such neighbours is also even. As a result the degree of Type-II vertices are also even. This proves the theorem. $\square$

3.2.2 Domination number, Chromatic Number and Perfectness of $\Gamma(D_n)$

In this section, we study the domination number, chromatic number of $\Gamma(D_n)$ and characterize when $\Gamma(D_n)$ is perfect.

Theorem 3.2.7 *The domination number of $\Gamma^*(D_n)$ is given by*

$$\gamma(\Gamma^*(D_n)) = \begin{cases} 1, & \text{if } n \text{ is a prime power,} \\ \pi(n) + 1, & \text{otherwise.} \end{cases}$$

Proof : If n is a prime power, by Proposition 3.2.5, $\Gamma^*(D_n)$ is a star and hence the theorem follows. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Clearly $\{\langle r\rangle, \langle r^{p_1}, s\rangle, \langle r^{p_2}, s\rangle, \ldots, \langle r^{p_k}$ is a dominating set of $\Gamma^*(D_n)$ and hence $\gamma(\Gamma^*(D_n)) \leq k + 1$.

If possible, let $S = \{x_1, x_2, \ldots, x_k\}$ be a dominating set of size k. Set $m = p_1 p_2 \cdots p_k$ and consider the set of $k+1$ vertices $A = \{\langle r^{m/p_1}\rangle, \langle r^{m/p_2}\rangle, \ldots, \langle r^{m/p_k}\rangle, \langle r^m, s\rangle$ Among these $k+1$ vertices, at least one of them is not in S. Without loss of generality, let $\langle r^{m/p_1}\rangle \notin S$ and $\langle r^{m/p_1}\rangle \sim x_1$. Then x_1 is of the form $\langle r^{p_1^{\beta_1}}, r^{i_1}s\rangle$. Note that x_1 is not adjacent to any one of k vertices in the set $A' = \{\langle r^{m/p_2}\rangle, \ldots, \langle r^{m/p_k}\rangle, \langle r^m, s\rangle\}$. By similar argument, not all of these k vertices in A' belong to S. Without loss of generality, let $\langle r^{m/p_2}\rangle \notin S$ and $\langle r^{m/p_1}\rangle \sim x_2$. Proceeding similarly, we get $x_2 = \langle r^{p_2^{\beta_2}}, r^{i_2}s\rangle, \ldots, x_k = \langle r^{p_k^{\beta_k}}, r^{i_k}s\rangle$.

If n is odd, then $\langle r^m, s\rangle$ is not adjacent to any x_i, a contradiction. If n is even, then either $\langle r^m, s\rangle$ or $\langle r^m, rs\rangle$ is not dominated by any x_i, a contradiction. Hence, $\gamma(\Gamma^*(D_n)) = k+1$. □

Theorem 3.2.8 $\Gamma(D_n)$ *is weakly perfect, i.e., the clique number and chromatic number of* $\Gamma(D_n)$ *are given by*

$$\chi(\Gamma(D_n)) = \omega(\Gamma(D_n)) = \begin{cases} \pi(n)+1, & \text{if } n \text{ is odd} \\ \pi(n)+2, & \text{if } n \text{ is even.} \end{cases}$$

Proof : We first deal with the case when n is odd, say $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are distinct odd primes. Consider the set $A = \{\langle r\rangle, \langle r^{p_1}, s\rangle, \langle r^{p_2}, s\rangle, \ldots, \langle r^{p_k}, s$ Clearly A forms a clique of size $k+1 = \pi(n)+1$, i.e., $\omega(\Gamma(D_n)) \geq \pi(n)+1$. Let M be a maximum clique of $\Gamma(D_n)$ of size $t \geq k+2$. If M contains only vertices of Type-II, then $M \cup \langle r\rangle$ is a clique properly containing M, a contradiction. Thus M always contains a vertex of Type-I. As no two vertices of Type-I are adjacent, M can have exactly one vertex of Type-I. Without loss of generality, we can assume the Type-I vertex in M to be $\langle r\rangle$. Let $M = \{\langle r\rangle, \langle r^{a_1}, r^{b_1}s\rangle, \langle r^{a_2}, r^{b_2}s\rangle, \ldots, \langle r^{a_{t-1}}, r^{b_{t-1}}s\rangle\}$. Thus a_1, a_2, a_{t-1} are mutually coprime factors of n and $a_i \neq 1$. But as n has $\pi(n)$ distinct prime factors, it can have atmost $\pi(n) = k < t-1$ mutually coprime factors. Thus $\omega(\Gamma(D_n)) = \pi(n) + 1$.

Similarly, if n is even, i.e., $n = 2^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, it can be easily checked that $B = \{\langle r\rangle, \langle r^2, s\rangle, \langle r^2, rs\rangle, \langle r^{p_2}, s\rangle, \ldots, \langle r^{p_k}, s\rangle\}$ is a clique of size $k+2 = \pi(n) + 2$. Thus $\omega(\Gamma(D_n)) \geq \pi(n) + 2$. Let M be a maximum clique of $\Gamma(D_n)$ of size t. As in the previous case, M have exactly one vertex of Type-I. Let $M = \{\langle r\rangle, \langle r^{a_1}, r^{b_1}s\rangle, \langle r^{a_2}, r^{b_2}s\rangle, \ldots, \langle r^{a_{t-1}}, r^{b_{t-1}}s\rangle\}$. Arguing as in the previous case, the number of odd divisors of n among a_1, a_2, a_{t-1} is atmost $k-1$. Again due to the adjacency condition of Type-II vertices, the number of odd divisors of n among a_1, a_2, a_{t-1} is atmost 2. Thus M can have atmost $1 + 2 + (k-1) = k+2$ vertices, i.e., $\omega(\Gamma(D_n)) = \pi(n) + 2$.

As $\chi \geq \omega$, it suffices to produce a proper colouring using ω colours. If $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ is odd, define

$$A_1 = \{\langle r^d\rangle, \langle r^d, r^i s\rangle : p_1|d\},\ A_2 = \{\langle r^d\rangle, \langle r^d, r^i s\rangle : p_2|d\} \setminus A_1, \cdots,$$

$$A_j = \{\langle r^d\rangle, \langle r^d, r^i s\rangle : p_j|d\} \setminus \bigcup_{l=1}^{j-1} A_l,\ \text{where } j = 1, 2, \ldots, k.$$

Observe that $A_1, A_2, \ldots, A_k$ are independent sets in $\Gamma(D_n)$. We assign the colour j to all the vertices in A_j and the $k+1$ the colour to $\langle r\rangle$. It can be easily checked that this is a proper colouring of $\Gamma(D_n)$ using $k+1=\pi(n)+1$ colours.

Similarly, if n is even, we construct similar independent sets for each prime as above, with the following exception for the prime 2. For the prime 2, we construct two sets $X=\{\langle r^d\rangle,\langle r^d,r^is\rangle : 2|d, i \text{ is odd}\}$ and $Y=\{\langle r^d\rangle,\langle r^d,r^is\rangle : 2|d, i \text{ is even}\}$. One can easily check that this gives a proper colouring $\Gamma(D_n)$ using $\pi(n)+2$ colours. □

Theorem 3.2.9 $\Gamma(D_n)$ *is perfect if and only if one of the two conditions hold:*

- *n is odd and $\pi(n)\leq 4$.*
- *n is even and either $\pi(n)\leq 2$ or $\pi(n)=3$ and $4\nmid n$.*

Proof : If n is odd and $\pi(n)\geq 5$, let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_5^{\alpha_5}m$, where p_i's are odd primes which are coprime to m. Then $\langle r^{p_1p_2},s\rangle\sim\langle r^{p_3p_4},s\rangle\sim\langle r^{p_2p_5},s\rangle\sim\langle r^{p_1p_4},s\rangle\sim\langle r^{p_3p_5},s\rangle\sim\langle r^{p_1p_2},s\rangle$ is an induced 5-cycle in $\Gamma(D_n)$ and hence $\Gamma(D_n)$ is not perfect.

Let n be odd and $\pi(n)\leq 4$. Let $C: x_1\sim x_2\sim\cdots\sim x_{2t+1}\sim x_1$ be an induced odd cycle in $\Gamma(D_n)$. As n is odd and any subgroup of D_n is of the form $\langle r^d\rangle$ or $\langle r^d,r^is\rangle$, it follows from the adjacency condition that $\langle r^{d_1}\rangle\sim\langle r^{d_2},r^is\rangle$ or $\langle r^{d_1},r^is\rangle\sim\langle r^{d_2},r^js\rangle$ if and only if $gcd(d_1,d_2)=1$. Thus for each vertex x_i in C we can associate a factor d_i of n such that $x_i\sim x_j$ if and only if $gcd(d_i,d_j)=1$. Now, by following the steps in the proof of Theorem 3.2 in [39], one can show that $\Gamma(D_n)$ is perfect.

If n is even and $\pi(n) \geq 4$, let $n = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_4^{\alpha_4} m$, where p_i's are odd primes which are coprime to m. Then $\langle r^{p_2} \rangle \sim \langle r^{2p_2p_3}, rs \rangle \sim \langle r^{p_3p_4} \rangle \sim \langle r^{2p_4}, r^2s \rangle \sim \langle r^{2p_2}, s \rangle \sim \langle r^{p_2} \rangle$ is an induced 5-cycle in the complement of $\Gamma(D_n)$ and hence $\Gamma(D_n)$ is not perfect.

If $\pi(n) = 3$ and $4|n$, let $n = 2^{\alpha} p_2^{\alpha_2} p_3^{\alpha_3}$ where p_i's are odd primes. Then $\langle r^{p_1} \rangle \sim \langle r^4, s \rangle \sim \langle r^{p_2} \rangle \sim \langle r^{2p_1}, s \rangle \sim \langle r^{4p_2}, rs \rangle \sim \langle r^{p_1} \rangle$ is an induced 5-cycle in the complement of $\Gamma(D_n)$ and hence $\Gamma(D_n)$ is not perfect.

Thus, if n is even, we are left with two cases, either $n = 2^{\alpha} p_2^{\alpha_2}$ or $n = 2p_2^{\alpha_2} p_3^{\alpha_3}$. These two cases are dealt with in the following two lemmas. $\square$

Lemma 3.2.6 *If $n = 2^{\alpha} p_2^{\alpha_2}$, then $\Gamma(D_n)$ is perfect.*

Proof : Note that any vertex of the form $\langle r^d \rangle$ or $\langle r^d, r^i s \rangle$ where $2p_2|d$ are of degree 0 or 2 respectively in $\Gamma(D_n)$. In fact, $\langle r^d, r^i s \rangle$ is adjacent to exactly two vertices, namely $\langle r \rangle$ and exactly one of $\langle r^2, s \rangle$ and $\langle r^2, rs \rangle$. If possible, let $C : x_1 \sim x_2 \sim \cdots \sim x_{2t+1} \sim x_1$ be an induced odd cycle of length atleast 5 in $\Gamma(D_n)$. Clearly C must have atleast one Type-II vertex. As $\langle r \rangle$ does not lie on C, any vertex of the form $\langle r^d, r^i s \rangle$ where $2p_2|d$ does not lie on C.

Claim A: If $x_1 = \langle r^{d_1}, r^i s \rangle$ is a Type-II vertex on C, then d_1 is even.

Proof of Claim A: If d_1 is odd, then $d_1 = p_2^{\beta}$. As $x_1 \nsim x_3, x_4$, we have $x_3 = \langle r^{p_2^a} \rangle$ or $\langle r^{p_2^a}, r^j s \rangle$ and $x_4 = \langle r^{p_2^b} \rangle$ or $\langle r^{p_2^b}, r^k s \rangle$. In any case, we have $x_3 \nsim x_4$, a contradiction.

Claim B: There exists no Type-I vertex on C.

Proof of Claim B: If there exists two vertices, say x_1, x_k of Type-I on C. Clearly they must be non-adjacent. Using Claim A, $x_1 = \langle r^{p_2^{\beta}} \rangle$ and $x_k =$

$\langle r^{p_2^{\beta'}} \rangle$. But as $x_2, x_{2t+1} \sim x_1$, we must have $x_k \sim x_2, x_{2t+1}$, a contradiction. So atmost one Type-I vertex can be on C, say $x_1 = \langle r^{p_2^{\beta}} \rangle$. As $x_1 \not\sim x_3, x_4$ and both are Type-II vertices, by Claim A, we must have $x_3 = \langle r^{d_3}, r^i s\rangle$ and $x_4 = \langle r^{d_4}, r^j s\rangle$ where $2p_2$ divides d_3 and d_4. However such vertices do not lie on C.

Thus all the vertices on C are of Type-II, i.e., $x_l = \langle r^{d_l}, r^{i_l} s\rangle$ for $l = 1, 2, \ldots, 2t+1$ where d_l's are even. Again from the adjacency condition, we have all of $i_1 - i_2, i_2 - i_3, \ldots, i_{2t+1} - i_1$ to be odd. Adding all of them, we get the sum of odd number of odd integers to be zero, a contradiction. Thus $\Gamma(D_n)$ has no induced odd cycle of length atleast 5. Similarly, it can be shown that $\Gamma(D_n)^c$ has no induced odd cycle of length atleast 5. Hence $\Gamma(D_n)$ is perfect. $\square$

Lemma 3.2.7 *If $n = 2^{\alpha} p_2^{\alpha_2} p_3^{\alpha_3}$, then $\Gamma(D_n)$ is perfect*

Proof : If possible, let $C : x_1 \sim x_2 \sim \cdots \sim x_{2t+1} \sim x_1$ be an induced odd cycle of length atleast 5 in $\Gamma(D_n)$. As no two Type-I vertices are adjacent, thus we must have atleast $t+1 \geq 3$ Type-II vertices in C.

Claim 1: $\langle r^d, r^i s\rangle$, where $2p_1p_2|d$ does not lie in C.

Proof of Claim 1: Its only neighbours are $\langle r\rangle$ and exactly one of $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$. As $\langle r\rangle$ is adjacent to all Type-II vertices and there are atleast 3 Type-II vertices in C, $\langle r\rangle$ does not lie on C. Thus $\langle r^d, r^i s\rangle$ can have atmost one neighbour in C, which is a contradiction as C is a cycle.

Claim 2: None of $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$ lie in C.

Proof of Claim 2: If $x_1 = \langle r^2, s\rangle$ lies in C, then as $\langle r^2, s\rangle$ is a maximal subgroup of index 2 in D_n, all of $x_3, x_4, \ldots, x_{2t}$ are contained in x_1. Thus, using Claim 1, without of loss of generality, we can assume that $x_3 = \langle r^{2p_1^{\beta_1}}, r^i s\rangle$ and $x_3 = \langle r^{2p_2^{\beta_2}}, r^j s\rangle$. As $x_3 \sim x_4$, we have $i - j$ is odd. On the other hand, as $x_1 \not\sim x_3, x_4$, we must have i and j to be both even. This contradicts the parity of $i - j$.

Claim 3: Vertices of the form $\langle r^{p_1^{\beta_1} p_2^{\beta_2}}\rangle$ and $\langle r^{p_1^{\beta_1} p_2^{\beta_2}}, r^i s\rangle$ do not lie in C.

Proof of Claim 3: As $\langle r^{p_1^{\beta_1} p_2^{\beta_2}}\rangle$ is adjacent only with $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$, the claim follows from Claim 2. Similarly, only neighbours of $\langle r^{p_1^{\beta_1} p_2^{\beta_2}}, r^i s\rangle$ in $\Gamma(D_n)$ are $\langle r\rangle$, $\langle r^2\rangle$ and exactly one of $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$. However, from Claim 2, its only possible neighbour in C is $\langle r^2\rangle$, a contradiction. Hence Claim 3 holds.

Claim 4: $\langle r^2\rangle$ lies in C.

Proof of Claim 4: Suppose $\langle r^2\rangle$ does not in C. Then from Claims 1,2 and 3, it follows that for any vertex $\langle r^{d_i}\rangle$ or $\langle r^{d_i}, r^i s\rangle$ in C, d_i must be of the form $p_1^{\beta_1}, p_2^{\beta_2}, 2p_1^{\beta_1}$ or $2p_2^{\beta_2}$. Again, as C is cycle, d_i must be alternately divisible by p_1 and p_2. But this contradicts that C is an odd cycle. Thus the claim follows.

Let $x_1 = \langle r^2\rangle$ be a vertex on C. As x_1 is a Type-I vertex, from the adjacency condition and previous claims, without loss of generality, we have $x_2 = \langle r^{p_1^{\beta}}, r^i s\rangle$ and $x_{2t+1} = \langle r^{p_1^{\beta'}}, r^j s\rangle$. Then x_3 must be of one of the 4 forms, namely $\langle r^{p_2^{\beta_2}}\rangle, \langle r^{p_2^{\beta_2}}, r^j s\rangle, \langle r^{2p_2^{\beta_2}}\rangle$ and $\langle r^{2p_2^{\beta_2}}, r^j s\rangle$. However, in any case, we have $x_3 \sim x_{2t+1}$, a contradiction. Thus $\Gamma(D_n)$ has no induced odd cycle of length atleast 5. $\square$

3.2.3 Isomorphisms of $\Gamma(D_n)$

In this section, we discuss some isomorphism results of $\Gamma(D_n)$. The first result (Theorem 3.2.10) shows that co-maximal graph of D_n uniquely determines n. The second result (Theorem 3.2.11) is more general in nature. It shows that nilpotent dihedral groups are uniquely determined by their comaximal subgroup graphs.

Lemma 3.2.8 *Let n and m be two positive integers such that $\Gamma(D_n) \cong \Gamma(D_m)$. Then n and m are of same factorization type.*

Proof : As $\Gamma(D_n) \cong \Gamma(D_m)$, from Theorem 3.2.4, it follows that n and m have same parity. Thus, by Theorem 3.2.8, $\pi(n) = \pi(m)$, i.e., m and n have same number of distinct prime factors. So we assume that $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $m = q_1^{\beta_1} q_2^{\beta_2} \cdots q_k^{\beta_k}$.

Consider the Type-I vertices other than $\langle r \rangle$ in $\Gamma(D_n)$. Note that $\{\langle r^{p_1} \rangle, \langle r^{p_1^2} \rangle, \cdots, \langle r^{p}$ is one of the twin class of size α_1. Similarly, we get twin classes of size $\alpha_2, \alpha_3, \ldots, \alpha_k$. Again, note $\{\langle r^{p_1 p_2} \rangle, \langle r^{p_1^2 p_2} \rangle, \cdots, \langle r^{p_1^{\alpha_1} p_2^{\alpha_2}} \rangle\}$ is a twin class of size $\alpha_1 \alpha_2$. Proceeding this way, Type-I vertices other than $\langle r \rangle$, can be partitioned into twin classes of size

$$\mathcal{P}_n = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_1\alpha_2, \alpha_2\alpha_3, \ldots, \alpha_1\alpha_2 \cdots \alpha_k\}.$$

Similarly for $\Gamma(D_m)$, we get

$$\mathcal{P}_m = \{\beta_1, \beta_2, \ldots, \beta_k, \beta_1\beta_2, \beta_2\beta_3, \ldots, \beta_1\beta_2 \cdots \beta_k\}.$$

As $\Gamma(D_n) \cong \Gamma(D_m)$, we have $\mathcal{P}_n = \mathcal{P}_m$. If $\alpha_i = \beta_{\sigma(i)}$ for some $\sigma \in S_k$, we are done. If no α_i is equal to any β_j, then without loss of generality, let $\alpha_1 = \min\{\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_k\}$. Therefore, $\alpha_1 < \beta_i$ for all i. Thus $\alpha_1 \in \mathcal{P}_n$, but $\alpha_1 \in \mathcal{P}_m$, as $\beta_i > \alpha_1$. This contradicts the fact $\mathcal{P}_n = \mathcal{P}_m$. Thus some α_i's are equal to some β_j. By suitable renaming, let $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_i = \beta_i$ and none of $\alpha_{i+1}, \ldots, \alpha_k$ is not equal to any of $\beta_{i+1}, \ldots, \beta_k$. Therefore each of $\alpha_{i+1}, \ldots, \alpha_k$ is product of atleast two β_j's. Similarly, each of $\beta_{i+1}, \ldots, \beta_k$ is product of atleast two α_j's.

We remove all the terms involving $\alpha_1, \alpha_2, \ldots, \alpha_i$ from $\mathcal{P}_n$ to get a new set $\mathcal{P}'_n$. Similarly, we remove all the terms involving $\beta_1, \beta_2, \ldots, \beta_i$ from $\mathcal{P}_m$ to get a new set $\mathcal{P}'_m$. Hence we have $\mathcal{P}'_n = \mathcal{P}'_m$.

Let $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_t}$ be the smallest element of $\mathcal{P}'_n$. Then at least one of $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_t}$ does not belong to $\{\alpha_1, \alpha_2, \ldots, \alpha_i\}$. Let $\alpha_{i_1} \notin \{\alpha_1, \alpha_2, \ldots, \alpha_i\}$. Then $\alpha_{i_1} \in \mathcal{P}'_n$ and $\alpha_{i_1} \leq \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_t}$. Thus α_{i_1} is also smallest in $\mathcal{P}'_n = \mathcal{P}'_m$.

Therefore $\alpha_{i_1} = \beta_{j_1}\beta_{j_2}\cdots\beta_{j_t} \in \mathcal{P}'_m$. Arguing similarly, without loss of generality, β_{j_1} is the smallest element in $\mathcal{P}'_m$. Thus $\alpha_{i_1} = \beta_{j_1}$, a contradiction. Hence, $\alpha_i = \beta_{\sigma(i)}$ for some $\sigma \in S_k$ and the theorem follows. □

Theorem 3.2.10 *Let n and m be two positive integers such that $\Gamma(D_n) \cong \Gamma(D_m)$. Then $n = m$.*

Proof : From Lemma 3.2.8, we get that $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ and $m = q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_k^{\alpha_k}$. Thus, it suffices to show that $p_i = q_i$ for all i. We consider the case when both m and n are odd. The case when both m and n are even can be handled similarly.

Consider the maximum clique $A = \{\langle r\rangle, \langle r^{p_1}, s\rangle, \langle r^{p_2}, s\rangle, \ldots, \langle r^{p_k}, s\rangle\}$ of $\Gamma^*(D_n)$ as defined in the proof of Theorem 3.2.8. Note that it contains exactly one vertex of Type-I and k-vertices of Type-II. As $\Gamma(D_n) \cong \Gamma(D_m)$, under any isomorphism, A is mapped to a maximum clique B of $\Gamma^*(D_m)$. Without loss of generality,

$$B = \{\langle r\rangle, \langle r^{q_1}, r^{i_1}s\rangle, \langle r^{q_2}, r^{i_2}s\rangle, \ldots, \langle r^{q_k}, r^{i_k}s\rangle\}.$$

Now, consider the number of Type-I and Type-II neighbours of Type-II vertices in A. For example, $\langle r^{p_i}, s\rangle$ has $(\tau(n/{p_i}^{\alpha_i}) - 1)$ many Type-I neighbours and $(\sigma(n/{p_i}^{\alpha_i}) - 1)$ many Type-II neighbours in $\Gamma^*(D_n)$. Similarly, we can compute the number of Type-I and Type-II neighbours of Type-II vertices in B. As $\Gamma(D_n) \cong \Gamma(D_m)$, the following two sets consisting of ordered pairs are equal.

$$\{(\tau(n/{p_1}^{\alpha_1}), \sigma(n/{p_1}^{\alpha_1})), (\tau(n/{p_2}^{\alpha_2}), \sigma(n/{p_2}^{\alpha_2})), \ldots, (\tau(n/{p_k}^{\alpha_k}), \sigma(n/{p_k}^{\alpha_k}))\}$$

$$= \{(\tau(m/{q_1}^{\alpha_1}), \sigma(m/{q_1}^{\alpha_1})), (\tau(m/{q_2}^{\alpha_2}), \sigma(m/{q_2}^{\alpha_2})), \ldots, (\tau(m/{q_k}^{\alpha_k}), \sigma(m/{q_k}^{\alpha_k}))\}$$

Again, as $\tau(m) = \tau(n), \sigma(m) = \sigma(n)$ and τ, σ are multiplicative functions, we have

$$\{(\tau({p_1}^{\alpha_1}), \sigma({p_1}^{\alpha_1})), (\tau({p_2}^{\alpha_2}), \sigma({p_2}^{\alpha_2})), \ldots, (\tau({p_k}^{\alpha_k}), \sigma({p_k}^{\alpha_k}))\}$$

$$= \{(\tau({q_1}^{\alpha_1}), \sigma({q_1}^{\alpha_1})), (\tau({q_2}^{\alpha_2}), \sigma({q_2}^{\alpha_2})), \ldots, (\tau({q_k}^{\alpha_k}), \sigma({q_k}^{\alpha_k}))\}$$

As these two sets are equal, there exists i such that $(\tau({p_1}^{\alpha_1}), \sigma({p_1}^{\alpha_1})) = (\tau({q_i}^{\alpha_i}), \sigma({q_i}^{\alpha_i}))$, i.e., $\alpha_1 = \alpha_i$ and hence $\sigma({p_1}^{\alpha_1}) = \sigma({q_i}^{\alpha_1})$, i.e., $p_1 = q_i$. Similarly, it can be shown that set of prime factors of m and n are same and as a result, $m = n$. $\square$

Theorem 3.2.11 *Let G be a finite solvable group such that $\Gamma(G) \cong \Gamma(D_{2^\alpha})$. Then $G \cong D_{2^\alpha}$.*

Proof : As $\Gamma^*(D_{2^\alpha})$ has a unique universal vertex, namely $\langle r \rangle$ and all other Type-I vertices are isolated, we get a subgroup H which is the unique universal vertex in $\Gamma^*(G)$.

Claim 1: H is a maximal subgroup G and $H \lhd G$.

Proof of Claim 1: If there exists a proper subgroup X of G such that $H \subsetneq X$, then $deg(H) \leq deg(X)$ in $\Gamma(G)$, a contradiction. Thus H is a maximal subgroup of G. If H is not normal in G, there exists $g \in G$ such that $H' = gHg^{-1} \neq H$. Note that $K \sim H$ if and only if $gKg^{-1} \sim gHg^{-1}$, i.e., $deg(H) = deg(H')$, a contradiction. Thus $H \lhd G$.

From Claim 1, it follows that G/H is a prime order group, i.e., $[G : H] = p$, for some prime p. Thus $|G| = p^a m$ and $|H| = p^{a-1}m$, where $p \nmid m$.

Claim 2: G is a group of prime power order.

Proof of Claim 2: Let q be a prime factor of m and K be a Sylow q-subgroup of G. If $K \nsubseteq H$, then $KH = G$, i.e.,

$$p^a m = \frac{(q^b)(p^{a-1}m)}{|H \cap K|} = \frac{(q^b)(p^{a-1}m)}{q^t} = q^{b-t}p^{a-1}m, \text{ i.e., } q^{b-t} = p, \text{ a contradiction.}$$

Thus if q is a prime factor of m, then every Sylow q-subgroup K of G is contained in H. Thus K corresponds to a Type-I vertex in $\Gamma(D_{2^\alpha})$ and hence, if $K \neq H$, then K is an isolated vertex in $\Gamma(D_{2^\alpha})$. However, as G is solvable, K has a Hall complement L of order $p^a m/q^b$ in G, i.e., $KL = G$, i.e., $K \sim L$. Thus either m has no prime factor, i.e., $m = 1$ or $K = H$. If $m = 1$, then G is p-group and the claim holds. If $K = H$, then $|H| = q^b$, i.e., $a = 1$ and $|G| = pq^b$.

Again, note that $\Gamma^*(D_{2^\alpha})$ has exactly two Type-II vertices of second highest degree, namely $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$ and every other Type-II vertices is adjacent to exactly one of $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$. Let K_1, K_2 be the two vertices in $\Gamma^*(G)$ corresponding to $\langle r^2, s\rangle$ and $\langle r^2, rs\rangle$ respectively. As H is the universal vertex in $\Gamma^*(G)$, we have $H \sim K_1$ and $H \sim K_2$, i.e., $K_1, K_2 \nsubseteq H$. Thus $|K_1| = pq^{t_1}$ and $|K_2| = pq^{t_2}$. Again, as $\langle r^2, s\rangle \sim \langle r^2, rs\rangle$, we have $K_1 \sim K_2$, i.e., $K_1K_2 = G$, i.e.,

$$pq^b = \frac{pq^{t_1} \cdot pq^{t_2}}{|K_1 \cap K_2|}, \text{ i.e., } |K_1 \cap K_2| = pq^{t_1+t_2-b}.$$

If $p \neq q$, $K_1 \cap K_2 \nsubseteq H$, i.e., $H \sim K_1 \cap K_2$ and $K_1 \cap K_2$ corresponds to a Type-II vertex. Hence, $K_1 \cap K_2$ must be adjacent to one of K_1 and K_2. However, $K_1 \cap K_2 \subseteq K_1, K_2$, this is a contradiction. Thus we must have $p = q$ and $|G| = p^{b+1}$. Hence Claim 2 holds. As G is a group of prime-power order, G is nilpotent and $\Gamma^*(G)$ has a unique universal vertex. Thus by Theorem 3.6 in [23], G must belong to one of the five families of groups, namely $3, 4, 5, 6, 7$. As $\Gamma^*(\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p)$ and $\Gamma^*(M_{p^n})$ has p many universal vertices, G is not isomorphic to $\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p$ or M_{p^n}. Again, as $\Gamma^*(SD_{2^n})$ a unique vertex of second highest degree, G is not isomorphic to SD_{2^n}. If $G \cong Q_{2^n}$, then number of isolated vertices in $\Gamma(G)$ is $n-2$ and the second highest degree is 2^{n-2}. However, $\Gamma(D_{2^\alpha})$ has $\alpha - 1$ isolated vertices and its second highest degree is 2^α. This is a contradiction and hence $G \ncong Q_{2^n}$. Hence $G \cong D_{2^{n-1}}$. Finally, comparing the number of isolated vertices, we get $G \cong D_{2^\alpha}$. □

3.3 Conclusion and Open Issues

In this chapter, we discussed various properties related to comaximal subgroup graph of $\mathbb{Z}_n$ and D_n. However, some of the isomorphism problems are yet to be answered and can be interesting topics of further research.

- If G is a finite group such that $\Gamma(G) \cong \Gamma(\mathbb{Z}_n)$, what can we say about G?
- For the same question pertaining to D_n, a partial answer is provided in Theorem 3.2.11. Although the general case is still open.