Chapter 4

Prime Ideal Sum Graph of a Commutative Ring

Recently, investigation of graphs associated to algebraic structures has become very common and investigating graph theoretic properties of rings attracted many researchers. So, many research papers has been emerged in which connections between algebraic properties of a ring and the graph-theoretic properties of its graph are studied (see for instance [3, 10, 13, 17] and [31])

It is known that behavior of ideals in a ring effects the structure of the ring and so ideals play crucial roles in the study of ring constructions. That is why it is useful to associate graphs to ideals of a ring as done for instance in [5, 4, 20] or in [35]. Studies on these topics in the literature motivate us to define a new graph containing ideals as vertices. So, the main purpose of this chapter is to introduce and study prime ideal sum graph associated with a ring.

Throughout, R is a commutative ring with unity and as usual the rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively. A prime ideal $\mathfrak{p}$ is said to be an *associated prime* of a commutative Noetherian ring R, if R has a non-zero element x such that $\mathfrak{p} = \text{Ann}(x)$. By $\text{Ass}(R)$ and $\text{Spec}(R)$ we mean the sets of all associated prime and prime ideals of R, respectively. A ring R is said to be a *general* $\text{ZPI} - ring$ if each of its ideal can be expressed as a finite direct product of prime ideals of R. Dedekind domians are indecomposable general ZPI-rings. A ring R is said to be *local* if it has a unique maximal left ideal and is said to be *reduced* if it has no non-zero nilpotent element. The set of all nilpotent elements of R is denoted by $\sqrt{0}$. For any undefined notation or terminology in ring theory, we refer the reader to [27, 29] and [42].

The main goal of this chapter is to introduce and study some of the basic properties of the *prime ideal sum graph* $PIS(R)$ of R [38]. The *prime ideal sum graph* of R is a graph whose vertices are non-zero proper ideals of R and two distinct vertices I and J are adjacent if and only if $I + J$ is a prime ideal of R. To avoid null graphs, throughout this chapter, we take R to be a commutative ring with unity with at least one non-zero proper ideal. We investigate the graph properties of $PIS(R)$ such as diameter, girth, domination number, etc. Among many results, we give in this section a necessary and sufficient condition for the completeness and connectedness of $PIS(R)$ (Theorems 4.1.3 and 4.1.4). In Section 4.4, we characterize $PIS(R)$ for a general ZPI-ring. We determine in this section diameter and girth of $PIS(R)$ under some assumptions on R (Theorems 4.4.2 and 4.4.3) and the idea of bipartition for $PIS(R)$ is also analyzed (Corollary 4.4.1).

4.1 Connectedness of *PIS(R)*

In this section we study some of the basic properties of prime ideal sum graph $PIS(R)$ of R like having universal and isolated vertices, completeness, connectedness, diameter, girth etc.

Theorem 4.1.1 *Let R be a ring. $PIS(R)$ has a universal vertex if and only if one of the two statements hold:*

1. *R is a local ring.*
2. *R has exactly two maximal ideals M_1 and M_2 such that $M_1 \cap M_2$ is a non-trivial minimal ideal and that there is no non-prime ideal properly containing $M_1 \cap M_2$.*

Proof : Let (1) hold, i.e., R be a local ring with unique maximal ideal M. Then for any ideal I of R, we have $I + M = M$, which is a prime ideal and hence $I \sim M$. Thus M is a universal vertex.

Let (2) hold and $I = M_1 \cap M_2$. Suppose J is a non-trivial ideal other than I and without loss of generality, let $J \subseteq M_1$. As I is a minimal ideal, it follows that $I \subsetneq I + J \subseteq M_1$. Now, from the given condition $I + J$ is a prime ideal and hence I is a universal vertex.

Conversely, let $PIS(R)$ have a universal vertex, say I. If R has a unique maximal ideal, the proof is done. Now, assume that R has at least three maximal ideals, say M_1, M_2 and M_3. Note that I can not be a maximal ideal as two distinct maximal ideals are not adjacent. As I is not a maximal ideal, then it is contained in some maximal ideals, say M_1, i.e., $I \subsetneq M_1$. If possible, let $I \not\subset M_2$. Then by maximality of M_2, $I + M_2 = R$. Hence $I \not\sim M_2$, a contradiction. Thus $I \subseteq M_1 \cap M_2 \cap M_3$. Now, two cases may occur.

Case 1: Let $I \neq M_1 \cap M_2 \cap M_3$. In this case, as $I + M_1 \cap M_2 \cap M_3 = M_1 \cap M_2 \cap M_3$ and I is a universal vertex in $PIS(R)$, $M_1 \cap M_2 \cap M_3$ is a prime ideal in R. But $M_1 M_2 M_3 \subseteq M_1 \cap M_2 \cap M_3$ implies that $M_1 = M_2 = M_3$, a contradiction.

Case 2: Let $I = M_1 \cap M_2 \cap M_3$. Let $J = M_1 \cap M_2$. Since intersection of two maximal ideals can not be prime, $I + J = J$ is not a prime ideal, i.e., $I \not\sim J$, which contradicts with our assumption that I is a universal vertex. Therefore, R has exactly two maximal ideals, say M_1 and M_2. By the same argument above, we conclude that $I = M_1 \cap M_2$.

Now, we show that I is a minimal ideal. If possible, let there exists a non-trivial ideal $J \subsetneq I = M_1 \cap M_2$. Then, as I is a universal vertex, $J + I = I = M_1 \cap M_2$ is prime, a contradiction. Thus I is a minimal ideal.

If possible, let J be a non-prime ideal such that $I = M_1 \cap M_2 \subsetneq J \subsetneq M_i$, where $i = 1$ or 2. But, this implies $I + J = J$, a non-prime ideal and hence $I \not\sim J$, a contradiction. Thus there does not exist such ideal J. □

Remark 4.1.1 *Let R be a ring which is not reduced. Then $\sqrt{0}$ is adjacent to each element of Spec(R).*

Please note that if R is a reduced ring then $\sqrt{0}$ is not a vertex of $PIS(R)$. So, the assumption R is not reduced in Remark 4.1.1 is not superfluous. In the following, we give some equivalent statements for that $\sqrt{0}$ is a universal vertex of $PIS(R)$.

Corollary 4.1.2 *Let R be a non-reduced ring. Then the following statements are equivalent:*

(1) Every element of R is either nilpotent or unit.

(2) R has a unique prime ideal.

(3) $(R, \sqrt{0})$ is a local ring.

(4) $\sqrt{0}$ is a universal vertex of $PIS(R)$.

Proof : (1) $\Rightarrow$ (2) It is clear by [42, 3.51 Exercise].
(2) $\Rightarrow$ (3) If R has a unique prime ideal, then it is $\sqrt{0}$ which is also a unique maximal ideal.
(3) $\Leftrightarrow$ (4) It follows from Theorem 4.1.1.
(3) $\Rightarrow$ (1) Let $(R, \sqrt{0})$ be a local ring. Suppose that $a \in R$ is non-nilpotent. Since $a \notin \sqrt{0}$ and $\sqrt{0}$ is a unique maximal ideal then a is unit. $\square$

Theorem 4.1.2 *An ideal I of R is an isolated vertex in $PIS(R)$ if and only if I is a maximal as well as minimal ideal of R.*

Proof : Let I be an isolated vertex in $PIS(R)$. If I is not a maximal ideal, then it is properly contained in a maximal ideal M and $I + M = M$. Thus

$I \sim M$ in $PIS(R)$, a contradiction, and so I is a maximal ideal. If I is not a minimal ideal, then there exists an ideal J of R such that $\{0\} \subsetneq J \subsetneq I$ and we have $I + J = I$, which is a prime ideal. Thus $I \sim J$, a contradiction, and so I is a minimal ideal of R.

Conversely, let I be a maximal as well as minimal ideal of R. If possible, assume that I is not isolated in $PIS(R)$. Then there exists a non-zero proper ideal J of R other than I such that $I + J$ is a prime ideal. If $J \not\subset I$, then as I is maximal, we have $I + J = R$, which is not prime. On the other hand if $J \subseteq I$, then by minimality of I, we have $J = \{0\}$ or $I = J$, a contradiction. Thus such an ideal J does not exist and hence I is an isolated vertex in $PIS(R)$. □

Next, we characterize rings whose prime ideal sum graph is complete.

Theorem 4.1.3 *$PIS(R)$ is complete if and only if R is a local ring and every proper non-prime ideal is a minimal ideal.*

Proof : Let $PIS(R)$ be a complete graph. Then $PIS(R)$ has a universal vertex and hence either of the two conditions holds by Theorem 4.1.1. Since two distinct maximal ideals can not be adjacent, R can not have two maximal ideals. Hence (1) in Theorem 4.1.1 holds. Let I be a non-zero proper ideal of R which is not prime. If possible, there exists an ideal J of R such that $\{0\} \subsetneq J \subsetneq I$, then $I + J = I$. Since I is not a prime ideal, $I \nsim J$, a contradiction to the completeness of $PIS(R)$. Thus every proper non-prime ideal is a minimal ideal.

Conversely, let R be a local ring with unique maximal ideal, say M, and every proper non-prime ideal be a minimal ideal. Let I_1 and I_2 be two distinct non-zero proper ideals of R. Then $I_1, I_2 \subseteq M$. Suppose $I_1 + I_2 = I_3 \subseteq M$ is not prime. Then by given condition, I_3 is a minimal ideal. But $I_1, I_2 \subseteq I_3$ and I_1, I_2 are non-zero ideal. Thus $I_1 = I_2 = I_3$, a contradiction. Thus $I_1 + I_2$ is prime and hence $I_1 \sim I_2$. Thus $PIS(R)$ is complete. $\square$

Theorem 4.1.4 *Let R be a ring. Then $PIS(R)$ is connected if and only if R is not a direct sum of two fields. If $PIS(R)$ is connected, then $diam(PIS(R)) \leq 4$. Moreover, if R is a principal ideal ring, then $diam(PIS(R)) \leq 2$.*

Proof : Let I_1, I_2 be two non-zero ideals in R. If $I_1 + I_2$ is a prime ideal, then $I_1 \sim I_2$ in $PIS(R)$. Otherwise, assume that $I_1 + I_2$ is not prime. Since R is a ring with unity, both I_1 and I_2 are contained in some maximal ideals of R. If they are contained in the same maximal ideal, say M, then we have $I_1 \sim M \sim I_2$, as maximal ideals are prime and hence $d(I_1, I_2) = 2$. Thus we assume that $I_1 \subset M_1, I_2 \subset M_2$ and $M_1 \neq M_2$ are two maximal ideals in R. Then we have a path of length 4 given by $I_1 \sim M_1 \sim M_1 \cap M_2 \sim M_2 \sim I_2$, unless $M_1 \cap M_2 = \{0\}$. Thus $d(I_1, I_2) \leq 4$ and hence $diam(PIS(R)) \leq 4$. However if $M_1 \cap M_2 = \{0\}$, we have $R \cong R/\{0\} \cong R/(M_1 \cap M_2) \cong R/M_1 \oplus R/M_2$. As M_1, M_2 are maximal ideals in R, R/M_i is a field. Thus R is a direct sum of two fields.

Conversely, if R is direct sum of two fields F_1 and F_2, the only non-trivial ideals of R are $\{0\} \times F_2$ and $F_1 \times \{0\}$. In this case, $PIS(R)$ consists of two isolated vertices and hence it is not connected. Thus the first part of the theorem is proved.

For the second part, when R is a principal ideal ring, if I_1 and I_2 are contained in same maximal ideal of R, then as it is shown in previous case, we have $d(I_1, I_2) \leq 2$. Thus, we assume that $I_1 \subset M_1, I_2 \subset M_2$ and $M_1 \neq M_2$ are two maximal ideals in R. Since R is a principal ideal ring, let $I_1 = (a), I_2 = (b), M_1 = (x)$ and $M_2 = (y)$, for some $a, b, x, y \in R$. First, we prove that $xy \neq 0$. Because if $xy = 0$, then $\{0\} = M_1M_2 = M_1 \cap M_2$, which implies that R is a direct sum of two fields and hence $PIS(R)$ is disconnected, a contradiction. Thus $xy \neq 0$. Let $J = (xy)$. Clearly $I_1 + J \subseteq M_1$. On the other hand, since $I_1 \not\subset M_2$, by maximality of M_2, we have $I_1 + M_2 = R$, i.e., $(a) + (y) = R$. Since $1 \in R$, there exist $u, v \in R$ such that $au + vy = 1$ which implies $aux + vxy = x$. Then $(a) + (xy) = I_1 + J$. Thus, $M_1 \subseteq I_1 + J$ and hence $I_1 + J = M_1$, which is a prime ideal and $I_1 \sim J$ in $PIS(R)$. Similarly, it can be shown that $I_2 + J = M_2$ and hence $I_2 \sim J$ in $PIS(R)$. Thus $I_1 \sim J \sim I_2$, $d(I_1, I_2) \leq 2$ and hence the theorem follows. □

Corollary 4.1.3 *If R is an integral domain, then $diam(PIS(R)) \leq 4$.*

4.2 Girth and Domination Number of *PIS(R)*

In this section, we study the girth and domination number of $PIS(R)$.

Theorem 4.2.1 *If any two non-comparable ideals are adjacent in $PIS(R)$,* *then*

$girth(PIS(R)) = 3$.

Proof : Let I_1 and I_2 be two non-comparable ideals which are adjacent in $PIS(R)$. Then $I_1 + I_2$ is a prime ideal in R. Since I_1 and I_2 are non-comparable, I_1+I_2 is strictly larger than both I_1 and I_2 and hence I_1, I_2, I_1+I_2 forms a triangle in $PIS(R)$, i.e., $girth(PIS(R)) = 3$. □

Corollary 4.2.1 *If $PIS(R)$ is acyclic or $girth(PIS(R)) > 3$, then no two non-comparable ideals of R are adjacent in $PIS(R)$ and adjacency occurs only in case of comparable ideals, i.e., for any edge in $PIS(R)$, one of the terminal vertices is a prime ideal of R.*

Theorem 4.2.2 *If $girth(PIS(R)) = n$, then there exist at least $\lfloor n/2 \rfloor$ distinct prime ideals in R.*

Proof : By Theorem 4.2.1, if two non-comparable ideals are adjacent in $PIS(R)$, then $girth(PIS(R)) = 3$ and the sum of those two non-comparable ideals forms a prime ideal, and hence R contains at least $\lfloor 3/2 \rfloor = 1$ prime ideal. Thus we assume that $girth(PIS(R)) > 3$, i.e., by Corollary 4.2.1, adjacency occurs only in case of comparable ideals. Let $I_1 \sim I_2 \sim I_3 \sim \cdots \sim I_n \sim I_1$ be a cycle of length n. First, we observe that neither $I_1 \subset I_2 \subset I_3 \subset \cdots \subset I_n \subset I_1$ nor $I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset I_1$ can hold, as in both the cases all the ideals will be equal. Thus, without loss of generality, we have $I_1, I_3 \subset I_2$ and I_2 is a prime ideal. Hence we have the following two cases:

Case I: Let $I_1, I_3 \subset I_2$ and $I_3 \subset I_4$. Then, we have I_2, I_4 to be prime ideals.
Case II: Let $I_1, I_3 \subset I_2$ and $I_4 \subset I_3$. Then, we have I_2, I_3 to be prime ideals.
In any case, we get at least 2 ideals to be prime in R among I_1, I_2, I_3 and I_4. Continuing in this manner till I_n, we get atleast $\lfloor n/2 \rfloor$ ideals which is prime in R. Hence, the theorem follows. $\square$

Corollary 4.2.2 *Let R has k prime ideals. Then, $PIS(R)$ is either acyclic or $girth(PIS(R)) \leq 2k$.*

Theorem 4.2.3 *Let $\mathcal{M}$ be the set of all maximal ideals of R. Then $\mathcal{M}$ is a minimal dominating set of $PIS(R)$ and $\gamma(PIS(R)) \leq |\mathcal{M}|$. Moreover, $\gamma(PIS(R)) = 1$ if and only if R is a local ring or R has exactly two maximal ideals M_1 and M_2 such that $M_1 \cap M_2$ is a non-trivial minimal ideal such that there is no non-prime ideal properly containing $M_1 \cap M_2$. Also, if R has exactly two maximal ideals which does not satisfy the above condition, then $\gamma(PIS(R)) = 2$.*

Proof : Since any ideal I of R is contained in some element M of $\mathcal{M}$ and $I + M = M$, which is a prime ideal, $\mathcal{M}$ dominates $PIS(R)$. Let $M \in \mathcal{M}$. It is to be observed that $\mathcal{M} \setminus \{M\}$ does not dominate M and hence fails to dominate $PIS(R)$. Thus $\mathcal{M}$ is a minimal dominating set of $PIS(R)$ and $\gamma(PIS(R)) \leq |\mathcal{M}|$. The second part follows from Theorem 4.1.1. For the third part, observe that $\gamma(PIS(R)) \leq 2$. However, it cannot be 1, by Theorem 4.1.1. Hence, $\gamma(PIS(R)) = 2$. $\square$

Remark 4.2.3 *The inequality in previous theorem may be strict. For example, consider $PIS(\mathbb{Z}_{30})$. It is isomorphic to C_6 given by $(2) \sim (6) \sim (3) \sim (15) \sim (5) \sim (10) \sim (2)$ with 3 maximal ideals $(2), (3), (5)$. However, domination number of the graph is 2, e.g., $\{(2), (15)\}$ forms a dominating set of $PIS(\mathbb{Z}_{30})$.*

4.3 Homomorphisms of *PIS(R)*

In this section, we discuss some properties related to homomorphisms and isomorphisms of $PIS(R)$.

Proposition 4.3.1 *Let $\varphi : R \to S$ be an onto ring homomorphism. Then $\psi : PIS(S) \to PIS(R)$ is a graph homomorphism where $\psi(I) = \varphi^{-1}(I)$ for any ideal I of S.*

Proof : Since $\varphi^{-1}(I)$ is an ideal of R whenever I is an ideal of S, and moreover φ is an onto ring homomorphism and I is a non-zero proper ideal of S, the map ψ is well-defined. Let $I \sim J$ in $PIS(S)$. Then $I + J$ is a prime ideal in S. Therefore $\psi(I + J) = \varphi^{-1}(I + J) = \varphi^{-1}(I) + \varphi^{-1}(J)$ is a prime ideal in R. Thus $\varphi^{-1}(I) \sim \varphi^{-1}(J)$ in $PIS(R)$, i.e., $\psi(I) \sim \psi(J)$ in $PIS(R)$. Hence $\psi : PIS(S) \to PIS(R)$ is a graph homomorphism. □

Corollary 4.3.2 *If R and S are two isomorphic commutative rings with unity, then $PIS(R)$ and $PIS(S)$ are isomorphic as graphs.*

Proof : It immediately follows from Proposition 4.3.1. □

The following example shows that there exist non-isomorphic rings whose corresponding prime ideal sum graphs are isomorphic.

Example 4.3.1 *Consider $\mathbb{Z}_{12}$ and $\mathbb{Z}_{18}$. Clearly, they are not isomorphic as rings. However, both of their corresponding prime ideal sum graphs are isomorphic to P_4.*

Theorem 4.3.1 *Let $\varphi : R \to S$ be an onto ring homomorphism. Then $\omega(PIS(S)) \leq \omega(PIS(R))$.*

Proof : Let $\mathcal{M}$ be a maximum clique in $PIS(S)$. We claim that $\psi(\mathcal{M})$ is a clique in $PIS(R)$, where ψ is as defined in Proposition 4.3.1. Let I_1, I_2 be two ideals of R in $\psi(\mathcal{M})$. Then there exists ideals J_1, J_2 of S in $\mathcal{M}$ such that $\psi(J_1) = I_1$ and $\psi(J_2) = I_2$. As $\mathcal{M}$ is a clique in $PIS(S)$, $J_1 \sim J_2$ in $PIS(S)$ and since ψ is a graph homomorphism, $I_1 \sim I_2$ in $PIS(R)$. Thus $\psi(\mathcal{M})$ is a clique in $PIS(R)$. Also as $\mathcal{M}$ is a clique, $|\mathcal{M}| = |\psi(\mathcal{M})|$. Thus, we have $\omega(PIS(R)) \geq |\psi(\mathcal{M})| = \omega(PIS(S))$. $\square$

Theorem 4.3.2 *Let R be a commutative ring with unity and I be an ideal of R. Let $PIS_I(R)$ be the subgraph of $PIS(R)$ induced by the ideals of R containing I. Then $PIS_I(R)$ and $PIS(R/I)$ are isomorphic as graphs.*

Proof : Let us define a function $\varphi : PIS_I(R) \to PIS(R/I)$ given by $\varphi(J) = J/I$. Clearly φ is a bijection. Now $J_1 \sim J_2$ in $PIS_I(R)$ implies that $J_1 + J_2$ is prime ideal in R containing I. Thus, $J_1/I + J_2/I = (J_1 + J_2)/I$ is a prime ideal in $PIS(R/I)$ and hence $\varphi(J_1) \sim \varphi(J_2)$ in $PIS(R/I)$. Similarly,

$J_1/I \sim J_2/I$ in $PIS(R/I)$ implies that $J_1/I + J_2/I = (J_1 + J_2)/I$ is a prime ideal in $PIS(R/I)$, i.e., $J_1 + J_2$ is prime ideal in R. Thus $J_1 \sim J_2$ in $PIS(R)$ and as a result, φ is a graph isomorphism. □

4.4 Prime Ideal Sum Graph of a General ZPI-Ring

In this section, the prime ideal graph $PIS(R)$ is studied for general ZPI-rings. Moreover, the graph properties of $PIS(\mathbb{Z}_n)$ such as diameter, girth and domination number are studied.

Theorem 4.4.1 *Let R be a general ZPI-ring which is not a field and $|\text{Ass}(R)| = 1$. Then $PIS(R)$ is a single vertex graph or a star graph.*

Proof : Let $\text{Ass}(R) = \{P\}$. Then $0 = P^k$ for some $k \geq 1$. Since R is not a field, then $k > 1$.

Case 1: Assume that $k = 2$. If I is a non-zero proper ideal of R, then $0 = P^2 \subsetneq I \subseteq P$. Hence we conclude that $I = P$ by [27, Theorem 39.2]. Thus the graph has a single vertex p.

Case 2: Assume that $k \geq 3$. Then by Theorem 4.1.1, $PIS(R)$, being a local ring, has P as a universal vertex. On the other hand, any two other ideals of R, say P^i and P^j with $i \neq j \neq 1$ are non-adjacent. Thus, $PIS(R)$ is a star graph. □

Theorem 4.4.2 *Let R be a principal ideal ring which is not a field. Then,*

$$diam(PIS(R)) = \begin{cases} 0, & \text{if } \operatorname{Ass}(R) = \{P\} \text{ and } 0 = P^2 \\ 1, & \text{if } \operatorname{Ass}(R) = \{P\} \text{ and } 0 = P^3,\ P^2 \neq 0 \\ \infty, & \text{if } \operatorname{Ass}(R) = \{P_1, P_2\} \text{ and } 0 = P_1P_2 \\ 2, & \text{otherwise} \end{cases}$$

Proof : As R is a principal ideal ring, $diam(PIS(R)) \leq 2$ by Theorem 4.1.4 except for $\operatorname{Ass}(R) = \{P_1, P_2\}$ and $0 = P_1P_2$, where the graph is disconnected. Now assume that $\operatorname{Ass}(R) = \{P\}$. If $0 = P^2$, then the graph contains a single vertex and hence has diameter 0. If $0 = P^3, P^2 \neq 0$, then the graph is isomorphic to a path on 2 vertices and hence its diameter is 1. For all other cases, to establish that the diameter is 2, it suffices to show the existance of non-adjacent vertices. If $|\operatorname{Ass}(R)| = 1$ and $0 = P^k$ for some $k > 3$, then the graph is a star graph by Theorem 4.4.1 and hence has diameter 2. If $|\operatorname{Ass}(R)| \geq 2$, say P_1, P_2 are two distinct associated primes, then P_1 and P_2 are two non-adjacent vertices. Thus the theorem follows. □

The following example shows that two associated primes can be adjacent unless R is a principal ideal ring.

Example 4.4.1 *Let* $R = K[x, y]/\langle x^2, xy\rangle$ *where* K *is a field and x,y are indeterminates. Then, the primary decomposition of* O_R *is* $\langle x^2,\, y\rangle = \langle x\rangle \cap \langle x^2, xy, y^n\rangle$ *and* $\operatorname{Ass}(R) = \{\langle x\rangle, \langle x, y\rangle\}$. *Since* $\langle x\rangle \subset \langle x, y\rangle$, *they are adjacent vertices.*

Theorem 4.4.3 *Let R be a general ZPI-ring which is not a field. Then girth$(PIS(R))$ is either 3 or ∞.*

Proof : If $|\text{Ass}(R)| = 1$, then $PIS(R)$ is a single graph or a star graph by Theorem 4.4.1 and so it has no cycle. Assume that $|\text{Ass}(R)| \geq 2$. If $\text{Ass}(R) = \{P_1, P_2\}$ and $0 = P_1P_2$, then the graph consists of two isolated vertices. So, without loss of generality, we may assume that P_1^{α} is a factor of 0 where $\alpha \geq 2$. Then P_1, P_1^2 and P_1P_2 forms a triangle in $PIS(R)$. Thus girth$(PIS(R)) = 3$. □

Corollary 4.4.1 *Let R be a general ZPI-ring which is not a field. Then the following statements are equivalent:*

(1) $PIS(R)$ is a bipartite graph.

(2) Either $\text{Ass}(R) = \{P\}$ and $P^2 \neq 0$ or $\text{Ass}(R) = \{P_1, P_2\}$ and $P_1P_2 = 0$

.

Proof : It follows from the proofs of Theorems 4.4.1 and 4.4.3. □

Theorem 4.4.4 *Let R be a general ZPI-ring which is not a field. Let $0 = P_1P_2 \cdots P_k$. , where $P_1, P_2, \ldots, P_k$ are distinct prime ideals of R, then $\omega(PIS(R)) = k$.*

Proof : We note that any clique in $PIS(R)$ can contain at most one prime ideal and if I_i and I_j be two vertices in a clique, then $I_i + I_j$ is prime. We claim that any clique of size k is either of the two types, ignoring the ordering of the primes:

- Type - I: $\{I_1, I_2, \ldots, I_k\}$ where $I_1 = P_1P_2, I_2 = P_1P_3, \ldots, I_{k-1} = P_1P_k$ and $I_k = P_1$.
- Type - II: $\{I_1, I_2, \ldots, I_k\}$ where $I_1 = P_1P_2, I_2 = P_1P_3, \ldots, I_{k-1} = P_1P_k$ and $I_k = P_2P_3 \cdots P_k$.

Proof of Claim: Let $S = \{I_1, I_2, \ldots, I_k\}$ be a clique of size k in $PIS(R)$.

Case I: Suppose that for all I_i, there exists a common prime factor, say P_1. Then $I_i = P_1J_i$ where $P_1 \nmid J_i$, $J_i|P_2 \cdots P_k$, J_i and J_j are coprime for all $i, j \in \{1, 2, \ldots, k\}$. Since the number of I_i's is k, therefore $J_i \in \{R, P_2, P_3, \ldots, P_k\}$. Thus S is of Type - I.

Case II: Suppose that $k - 1$ elements of S has a common prime factor. We assume, without loss of generality, that $P_1|I_i$ for $i \in \{1, 2, \ldots, k-1\}$ and $P_1 \nmid I_k$. Then $I_i = P_1J_i$ where $P_1 \nmid J_i$ and $J_i|P_2 \cdots P_k$, for $i \in \{1, 2, \ldots, k - 1\}$. Since $P_1 \nmid I_k$ and $I_i \sim I_k$ for all $i \in \{1, 2, \ldots, k-1\}$, $J_i \neq R$. Now, since the number of J_i's is $k - 1$, which is equal to the number of prime factors of 0 other than P_1, it is clear that $J_i \in \{P_2, P_3, \ldots, P_k\}$. Thus, ignoring the order of the primes, $I_1 = P_1P_2, I_2 = P_1P_3, \ldots I_{k-1} = P_1P_k$. Again, as I_k is adjacent to all the other I_i's and $P_1 \nmid I_k$, we have $I_k = P_2P_3 \cdots P_k$. Hence S is of Type - II.

Case III: Suppose that at most $k-2$ elements of S has a common prime factor, say P_1. We may assume that P_1 divides $I_1, I_2, \ldots, I_{k-2}$ and P_1 does not divide I_{k-1} and I_k. Hence $I_i = P_1J_i$ where $P_1 \nmid J_i$, $J_i|P_2 \cdots P_k$ and J_i and J_j are coprime for $i \in \{1, 2, \ldots, k - 2\}$. Thus each of $J_1, J_2, \ldots, J_{k-2}$ has a distinct prime factors from the set $\{P_2, P_3, \ldots P_k\}$. Without loss of generality, let $P_2|J_1, P_3|J_2, \ldots, P_{k-1}|J_{k-2}$.

Subcase III (a): If they are equal, i.e., $J_1 = P_2, J_2 = P_3, \ldots J_{k-2} = P_{k-1}$, then since $I_{k-1} \sim I_i$ for all $i \in \{1, 2, \ldots, k-2\}$ and $P_1 \nmid I_{k-1}$, we have $I_{k-1} = P_2 P_3 \cdots P_{k-1}$. But, by similar arguments, I_k is also equal to $P_2 P_3 \cdots P_{k-1}$, a contradiction.

Subcase III (b): Without loss of generality, let $J_1 = P_2 P_k, J_2 = P_3, J_3 = P_4, \ldots J_{k-2} = P_{k-1}$. Then $I_1 = P_1 P_2 P_k, I_2 = P_1 P_3, I_3 = P_1 P_4, \ldots, I_{k-2} = P_1 P_{k-1}$. Now, by using similar arguments as in Subcase III (a), we have $I_{k-1}, I_k \in \{P_2 P_3 \cdots P_{k-1}, P_3 \cdots P_{k-1} P_k\}$. Without loss of generality, let $I_{k-1} = P_2 P_3 \cdots P_{k-1}$ and $I_k = P_3 \cdots P_{k-1} P_k$. However, in this case, $I_{k-1} \nsim I_k$ as $I_{k-1} + I_k$ is not a prime, a contradiction.

Similar arguments hold, if for any $s < k-2$ many I_i's share a common factor and the rest do not. Thus Case III can not occur in a clique and hence any clique of size k is either of Type - I or Type - II, thereby proving the claim.

Now, suppose S is a clique of size t, which is greater than k. Then it has a sub-clique S' of size k, and by the above claim, S' is one of the above two types. Suppose S' is of Type - I, i.e., $S' = \{I_1, I_2, I_3, \ldots, I_k\}$ where $I_1 = P_1 P_2, I_2 = P_1 P_3, \ldots, I_{k-1} = P_1 P_k$ and $I_k = P_1$. Since $I_j \sim P_1$ for all j with $k+1 \leq j \leq t$, we have $P_1|I_j$, i.e, $I_j = P_1 J_j$ for all j with $k+1 \leq j \leq t$. However, since $I_j = P_1 J_j \sim P_1 J_i$ for all $k+1 \leq j \leq t$ and for all $i = 2, 3, \ldots, k$. We have $P_i \nmid J_j$ for all $i \in \{2, 3, \ldots, k\}$. Since P_1^2 is not a factor of 0, we have $P_1 \nmid J_j$. Thus $J_j = R$, i.e., $I_j = P_1$, a contradiction.

Thus S' is not of Type - I.

Assume that S' is of Type - II, i.e., $S' = \{I_1, I_2, I_3, \ldots, I_k\}$ where $I_1 = P_1P_2, I_2 = P_1P_3, \ldots, I_{k-1} = P_1P_k$ and $I_k = P_2P_3 \cdots P_k$. If $P_1|I_{k+1}$, by similar arguments as above, we get $I_{k+1} = P_1$, a contradiction, as $I_{k+1} \sim I_k$. Suppose, $P_i|I_{k+1}$ for some $i \in \{2, 3, \ldots, k\}$, say P_2. Then $I_{k+1} = P_2J_{k+1}$. Since $I_{k+1} \sim I_k = P_2P_3 \cdots P_k$, we have $P_3, P_4, \ldots, P_k \nmid J_{k+1}$. Since P_2^2 is not a factor of 0, we have $P_2 \nmid J_{k+1}$. Also we have shown that $P_1 \nmid I_{k+1}$. Hence $J_{k+1} = R$ and so $I_{k+1} = P_2$. But this implies that $I_{k+1} \nsim I_2$, a contradiction. Thus S' is not of Type - II.

Thus, there does not exist any clique of size greater than k and hence $\omega(PIS(R))$ is k. □

Theorem 4.4.5 *Let R be a general ZPI-ring which is not a field. Let $0 = {P_1}^{\alpha_1}{P_2}^{\alpha_2} \cdots {P_k}^{\alpha_k}$, where $P_1, P_2, \ldots, P_k$ are distinct prime ideals of R and $\alpha_i \geq 2$ for at least one $i \in \{1, 2, \ldots, k\}$, then $\omega(PIS(R)) = k + 1$.*

Proof : We claim that any clique of size $k+1$ is of the following type where α'_i denotes a positive integer less than or equal to α_i for each $i \in \{1, 2, ..., k\}$ ignoring the ordering of the primes:

Type - III: $\{I_1, I_2, \ldots, I_k, I_{k+1}\}$ where $I_1 = P_1P_2^{\alpha'_2}$, $I_2 = P_1P_3^{\alpha'_3}, \ldots, I_{k-1} = P_1{P_k}^{\alpha'_k}$, $I_k = P_1$ and $I_{k+1} = {P_1}^{\alpha'_1}$ for some $\alpha'_1 \geq 2$.

Proof of Claim: Let $S = \{I_1, I_2, \ldots, I_k, I_{k+1}\}$ be a clique of size $k + 1$ in $PIS(R)$.

Case I: Assume that there is no common prime factor which divides all I_i's. We may assume that P_1 divides $I_1, I_2, \ldots, I_k$ and P_1 does not divide

I_{k+1}. Then $I_i = P_1^{\alpha_{t_i}{}'} J_i$ where $P_1 \nmid J_i$, $J_i | P_2{}^{\alpha_2} \cdots P_k{}^{\alpha_k}$, $\alpha'_{t_i} \leq \alpha_1$, J_i and J_j are coprime for all $i, j \in \{1, \ldots, k\}$. Hence $J_i \in \{R, P_2{}^{\alpha'_2}, \cdots, P_k{}^{\alpha'_k}\}$ where $0 < \alpha'_i \leq \alpha_i$ for each i. Observe that if P_1^2 divides both I_i and I_j for some $i, j \in \{1, 2, ..., k\}$, we have $I_i \nsim I_j$, a contradiction. Thus $P_1^2 | I_i$ holds for at most one $i \in \{1, ..., k\}$. Thus, ignoring the order of the primes, we conclude the following subcases:

Subcase I (a): Let $P_1^2 | I_k$. Then $I_1 = P_1 P_2{}^{\alpha'_2}, I_2 = P_1 P_3^{\alpha'_3}, \cdots, I_{k-1} = P_1 P_k{}^{\alpha_k{}'}, I_k = P_1^{\alpha'_1}$ for some $\alpha'_1 \geq 2$. Since $I_{k+1} \sim I_i = P_1 P_{i+1}{}^{\alpha'_{i+1}}$ for all $i \in \{1, \ldots, k-1\}$ and $P_1 \nmid I_{k+1}$, we have $I_{k+1} = P_2 P_3 \cdots P_k$. On the other hand, since $I_k = P_1^{\alpha'_1}$, we have $I_k \sim I_{k+1} = P_2 P_3 \cdots P_k$, a contradiction.

Subcase I (b): Let $P_1^2 \nmid I_k$ and $P_1^2 | I_i$ for some $i \in \{1, ..., k-1\}$, say I_1. Then $I_1 = P_1^{\alpha'_1} P_2^{\alpha'_2}, I_2 = P_1 P_3^{\alpha'_3}, \cdots, I_k = P_1 P_k{}^{\alpha'_k}, I_k = P_1$ for some $\alpha'_1 \geq 2$. Similar to Subcase I (a), we conclude again the same contradiction.

Subcase I (c): Let $P_1^2 \nmid I_i$ for all $i \in \{1, ..., k+1\}$. Then $I_1 = P_1 P_2{}^{\alpha'_2}, I_2 = P_1 P_3^{\alpha'_3}, \cdots, I_{k-1} = P_1 P_k^{\alpha'_k}, I_k = P_1$. Similar to Subcase I (a), we conclude again the same contradiction.

Similar arguments hold, if any $s(< k)$ many I_i's share a common prime factor and the rest do not. Thus, we assume that all the elements of S has a common prime factor. In Case II and III, we show that this common prime factor is one of the primes P_i of which the power $\alpha_i \geq 2$. Since $\alpha_i \geq 2$ for at least one $i \in \{1, 2, \ldots, k\}$, without loss of generality, we assume that $\alpha_1 \geq 2$.

Case II: Assume that $\alpha_i = 1$ and P_i is a common prime factor of S, say P_2. Then $I_i = P_2J_i$ where $P_2 \nmid J_i, J_i|{P_1}^{\alpha_1}{P_3}^{\alpha_3}\cdots{P_k}^{\alpha_k}$, J_i and J_j are coprime, for $i, j \in \{1, 2, \ldots, k+1\}$. Thus $J_i \in \{R, {P_1}^{\alpha'_1}, P_3^{\alpha'_3}, \cdots, {P_k}^{\alpha_k\prime}\}$ for $i \in \{1, 2, \ldots, k\}$; and so, ignoring the order of the primes, $I_1 = P_2{P_1}^{\alpha'_1}$, $I_2 = P_2{P_3}^{\alpha'_3}$, ..., $I_{k-1} = P_2{P_k}^{\alpha'_k}$, $I_k = P_2$. Since $I_i \sim I_{k+1} = P_2J_{k+1}$ for all $i \in \{1, 2, ..., k\}$, $P_i \nmid J_{k+1}$ for all $i \in \{1, 3, \ldots, k\}$. Since $J_{k+1}|{P_1}^{\alpha_1}{P_3}^{\alpha_3}\cdots{P_k}^{\alpha_k}$, we get $J_{k+1} = R$; and so $I_{k+1} = I_k$, a contradiction.

Case III: Suppose that $\alpha_i \geq 2$ and P_i is a common prime factor, say P_1. Then $I_i + I_j = P_1$ for all $i, j \in \{1, 2, \ldots, k+1\}$. Hence $I_i = P_1J_i$ where $J_i|{P_2}^{\alpha_2}\cdots{P_k}^{\alpha_k}$, J_i and J_j coprime for all $i, j \in \{1, 2, \ldots, k+1\}$. Since the number of J_i's is $k+1$, therefore, $J_i \in \{R, {P_1}^{\alpha'_1}, {P_2}^{\alpha'_2}, \cdots, {P_k}^{\alpha'_k}\}$ for each $i \in \{1, ..., k\}$. Thus, ignoring the order of the primes, $I_1 = P_1P_2^{\alpha'_2}, \ldots$, $I_{k-1} = P_1{P_k}^{\alpha'_k}$, $I_k = P_1$ and $I_{k+1} = {P_1}^{\alpha'_1}$. Thus S is of Type - III.

Hence, the claim is proved, i.e., any clique of size $k+1$ is of Type - III. Now, suppose S is a clique of size t, which is greater than $k+1$. Then it has a sub-clique S' of size $k+1$, and by the above claim, S' is of Type - III, i.e., $S' = \{I_1, I_2, \ldots, I_k, I_{k+1}\}$ where $I_1 = P_1P_2^{\alpha'_2}, \ldots, I_{k-1} = P_1{P_k}^{\alpha'_k}$, $I_k = P_1$ and $I_{k+1} = {P_1}^{\alpha'_1}$ for some $\alpha'_1 \geq 2$. Since $I_{k+2} \sim I_k$ and $I_{k+2} \sim I_{k+1}$, we have $P_1|I_{k+2}$ but $P_1^2 \nmid I_{k+2}$. Then $I_{k+2} = P_1J_{k+2}$ such that J_{k+2} is not divisible by all primes $P_1, ..., P_k$ as $I_{k+2} \sim I_i$ for all $i \in \{1, 2, \ldots, k-1\}$. Thus, $J_{k+2} = R$ which implies $I_{k+2} = I_k$, a contradiction. Consequently, there does not exist any clique of size greater than $k+1$ and hence $\omega(PIS(R)) = k+1$. □

In a view of Theorems 4.4.4 and 4.4.5, we conclude the following result.

Corollary 4.4.2 *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$ where $p_1, p_2, \ldots, p_k$ are distinct* *prime integers. Then* $\omega(PIS(\mathbb{Z}_n)) = \begin{cases} k, & \textit{if } n \textit{ is square-free} \\ k+1, & \textit{else} \end{cases}$

Theorem 4.4.6 *Let J be a general ZPI-ring which is not a field. Then,*

(1) $\omega, \chi \geq |\mathrm{Ass}(R)|$.

(2) $\gamma \leq |\mathrm{Ass}(R)|$.

Proof : Assume that $0 = P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_k^{\alpha_k}$, where $P_1, P_2, \ldots, P_k$ are associated primes.

(1) Consider the set of ideals $A = \{P_1, P_1P_2, \ldots, P_1P_k\}$. As $P_1 + P_1P_i = P_1 = P_1P_i + P_1P_j$ for $i \neq j$, A is a clique. Thus, $\omega \geq k$. Now, as $\chi \geq \omega$ we have $\chi \geq k$.

(2) Consider the set of ideals $S = \{P_1, P_2, \ldots, P_k\}$. We claim that S is a dominating set. Let I be a non-trivial ideal of R. Then, I has a prime factor, say P_i, and hence $I + P_i = P_i$; i.e. I is adjacent to P_i. Thus, S dominates $PIS(R)$.

□

Corollary 4.4.3 *Let n be a positive integer that is not prime. Then, the following statements hold:*

(1) $PIS(\mathbb{Z}_n)$ *is a star graph if and only if* $n = p^k$ *with* $k \geq 3$.

(2) The diameter of $PIS(\mathbb{Z}_n)$ *is 2 unless* $n = p^2, p^3, pq$. *For* $n = p^2, p^3, pq$ *the diameter of* $PIS(\mathbb{Z}_n)$ *is 0, 1 and* ∞*, respectively.*

(3) *girth*$(PIS(\mathbb{Z}_n)) = 3$ *unless* $n = p^k$ *or* pq.

(4) $PIS(\mathbb{Z}_n)$ *bipartite if and only if* $n = p^k$ *or* pq.

(5) *If* $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$, *then* $\omega(PIS(\mathbb{Z}_n)), \chi(PIS(\mathbb{Z}_n)) \geq k$ *and* $\gamma(PIS(\mathbb{Z}_n)) \leq k$.